# A proof of the Nielsen-Thurston classification of mapping class elements

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This proof will follow along the lines of [Mar16, Chapter 8], but will flesh out some of the details, as well as outlining potential subtleties. First, we'll outline a fact about the structure of geodesics laminations.

## Complementary regions of geodesic laminations

Consider a geodesic lamination $\lambda$ on a surface $S$. We're interested in what the connected components of $S \setminus \lambda$ look like. Let $T$ be a connected component of $S \setminus \lambda$. We consider the metric completion of $T$ with the path metric induced by the hyperbolic metric. This completion gives us a finite volume hyperbolic manifold with boundary, such that the boundaries are totally geodesic. If a boundary component is a closed geodesic, we know what it looks like, i.e. a circle. When the boundary component is an open geodesic, then it forms a spike, i.e. if we denote by $\gamma$ the boundary geodesic, $\gamma$ stays within a bounded distance of another geodesic boundary component, quite like the edges of ideal polygons do.


Figure 1: The complementary regions have spikes and possibly genus. This image was taken from [Bon01].

By doubling along the geodesic boundary, we can see that any such component must have some finite genus, and a finite number of spikes. Furthermore, the boundary of a complementary region which has positive genus must deformation retract onto a non-trivial simple closed geodesic.

## Understanding finite order and reducible mapping classes

Finite order mapping classes are, as the name suggests, finite order elements in the mapping class group. Reducible elements are those elements which leave some multicurve invariant. To understand how the mapping class group acts on the Teichmüller space, we need to understand how it acts on $\mathcal{ML}$. To that effect, we have the following theorem.

**Theorem 1.** *If $\varphi(\mu) = \mu$ for some $\varphi \in \mathrm{MCG}(S)$ and non-zero $\mu \in \mathcal{ML}$, then $\varphi$ is either reducible or finite order.*

*Proof.* Suppose that the measured lamination $\mu$ is not full, i.e. some complementary region is not an ideal polygon. Since we know exactly what the complementary regions can look like, we know that the boundary deformation retracts to simple closed loop $\gamma$, for each complementary region that's not an ideal polygon. Since the boundaries are fixed by $\varphi$, each such complementary region has a finite orbit under iterates of $\varphi$. Let $M$ be the multicurve given by the orbit of $\gamma$. It's clearly $\varphi$-invariant, which means $\varphi$ is reducible. The same proof also works $\varphi(\mu) = \lambda\mu$ for some $\lambda \neq 1$. This will be important later when we're analyzing Pseudo-Anosov mapping classes.

The second case happens when all the complementary regions are actually ideal polygons. All of these ideal polygon have finitely many sides, and there are finitely many of them. Furthermore, $\varphi$ permutes the complementary regions, and possibly reorders their edges, but we can take a sufficiently high power of $\varphi$ (which, by abuse of notation, we'll also call $\varphi$) to assume that $\varphi$ leaves each complementary regions and all the edges invariant. We claim that this actually means that $\varphi$ is the trivial mapping class.

Consider the preimage $\tilde{\mu}$ of $\mu$ in $\mathbb{H}^2$. Note that all the complementary regions are all still ideal polygons, and the lifts of complementary regions in $S$. We now also consider a lift $\tilde{\varphi}$ of the map $\varphi$. We can choose this lift so it leaves some complementary region of $\tilde{\mu}$ invariant. In particular, it leaves all the edges invariant, which means it fixes their vertices in $\partial\mathbb{H}^2$ fixed. Our goal will be to show that $\varphi$ leaves every leaf in $\tilde{\mu}$, invariant, which means it fixes a dense collection of points on $\partial\mathbb{H}^2$, and hence it's trivial. Note that something like this fails for Dehn twists*.

Consider a leaf $\gamma$ left invariant by $\tilde{\varphi}$, and let $\tau$ be a small arc transverse to it. We'll show that leaves in a small neighbourhood of $\gamma$ are also left invariant by $\tilde{\varphi}$. We can parameterize the leaves near $\gamma$ by the point they intersect $\tau$ at. Pick the leaves intersecting $\tau$ in $[0, c]$, where $0$ is the point where $\gamma$ intersects $\tau$. Under $\tilde{\varphi}$, this segment gets sent to some $[0, \tilde{\varphi}(c)]$. But we know that the transverse measure is preserved, which means that $c = \tilde{\varphi}(c)$†. We now know that if a leaf is fixed, that sufficiently close by leaves are also fixed. We also know that if an edge leaf of a complementary region is fixed, so is every other edge leaf of that complementary region. From this, we see that every leaf in the lamination is fixed, whose endpoints are dense in $\partial\mathbb{H}^2$, which means $\tilde{\varphi}$ fixes the boundary, and hence is the trivial mapping classes. □

## Pseudo-Anosov mapping classes

We now define (after already having mentioned them a few times) Pseudo-Anosov mapping classes. We define them simply to be those mapping classes which aren't reducible or finite order. By Theorem 1, we have that the Pseudo-Anosov mapping classes act freely on $\mathcal{ML}$. We also know that the mapping class group acts continuously on $\overline{T_g} = T_g \cup \mathbb{P}\mathcal{ML}$. By the Brouwer fixed-point theorem, even a Pseudo-Anosov element must fix some point in $\overline{T_g}$. It certainly can't fix any point in $T_g$, otherwise it would turn out to be finite order, which means it fixes some point of $\mathbb{P}\mathcal{ML}$, i.e. there exists some measured lamination $\mu$ such that $\varphi(\mu) = \lambda\mu$. We know by Theorem 1 that $\lambda \neq 1$. Without loss of generality, we can assume $\lambda > 0$, and we'll call relabel the lamination $\mu_u$ and call it the unstable lamination. We'll show now that $\varphi$ scales another lamination down by $\lambda^{-1}$. The point corresponding to that in $\mathbb{P}\mathcal{ML}$ is the only other fixed point of $\varphi$. In this sense, Pseudo-Anosov elements are the analogs of hyperbolic elements in $\mathrm{PSL}(2, \mathbb{R})$.

**Theorem 2.** *Let $\varphi$ be a Pseudo-Anosov mapping class. Then there are two measured laminations $\mu_u$ and $\mu_s$ such that $\varphi(\mu_u) = \lambda\mu_u$ and $\varphi(\mu_s) = \lambda^{-1}\mu_s$ for some $\lambda > 1$. These laminations are full and minimal, and together they fill $S_g$.*

*Proof.* We've already outlined why we must have at least one such lamination $\mu_u$. We now show why $\mu_u$ must be minimal and full. We in fact show that every minimal component of $\mu_u$ must be full. This of course proves that $\mu_u$ must be minimal, since if it wasn't we'd have two disjoint full geodesic laminations, which isn't possible, since full laminations intersect every other lamination. That means if $\mu_u$ isn't a minimal lamination, none of its minimal components must be full, i.e. each minimal component has a complementary region that has genus. For simplicity, assume there are two minimal components $\mu_1$ and $\mu_2$, and let $S_1$ and $S_2$ be a complementary region for $\mu_1$ and $\mu_2$ respectively which isn't an ideal polygon. Then we know that the boundary of $S_i$ deformation retracts onto a non trivial curve. In fact, since the boundaries of $S_1$ and $S_2$ are disjoint, we can deformation retract them

*Although it is true that a Dehn twist $\varphi$ fixes a curve $\gamma$, it's not true that the corresponding lift $\tilde{\varphi}$ to $\mathbb{H}^2$ fixes every lift of $\gamma$. It only fixes one, based on our choice of lift, and its action on $\partial\mathbb{H}^2$ is a hyperbolic Möbius transformation, with axis the lift fixed by $\tilde{\varphi}$

†This is the point where the same proof no longer works for Pseudo-Anosov mapping classes.

onto curves whose intersection number is $0$. Varying this over $S_i$ and $S_j$, we find a multicurve preserved by $\varphi$, which contradicts the fact that it's Pseudo-Anosov. We thus see that $\mu_u$ must be minimal and full.

After replacing $\varphi$ with a high enough power, and lifting to the universal cover, we may assume the lift preserves a complementary polygonal region $R$. The vertices of $R$ in $\partial\mathbb{H}^2$ are fixed points of the lift $\tilde{\varphi}$. Consider the sides $s_1$ through $s_k$ of $R$. Since the lamination $\mu_u$ is minimal, these leaves must be dense in $\tilde{\mu_u}$, which means we can find a sequence of leaves in $\tilde{\mu_u}$ which approach each $s_i$. None of these leaves share an endpoint with $s_i$, because if they did, that would violate Lemma 8.3.8 in [Mar16], since we'd end with three geodesics incident on a single boundary point. Furthermore, we also know that the vertices of $R$ are local attractors for $\tilde{\varphi}$, since the pushforward of the measure increases the mass.

Now consider the map $\tilde{\varphi}$ restricted to the arc between two consecutive vertices of $R$. The map must have a fixed point in the interior. If it had two fixed points in the interior, we'd have an invariant box, which means its measure stays the same, but the contradicts the fact that $\tilde{\varphi}$ increases measure by a factor of $\lambda$. This means we have only one fixed point, and it must be a repelling fixed point.

Consider the geodesic lamination given by taking geodesics going from the repulsive fixed points, and take the closure of its $\Gamma$ orbit. That is another geodesic lamination, say $\mu_s$ preserved by $\varphi$, which means it must be minimal and full. The next thing we need to show is that $\mu_s$ admits a projective transverse measure left invariant by $\varphi$. Consider a lift of a closed geodesic close the attracting points of $\varphi^{-1}$[‡]. Looking at the Dirac mass $\gamma$ on that closed geodesic as an element of $\mathbb{P}\mathcal{ML}$, we see that some subsequence of $\phi^{-k}([\gamma])$ converges with support exactly equal to $\mu_s$. Consider all such measures, with the additional property that they have zero self intersection. That means they all lie in $\mathcal{ML}$, and form a convex cone, which means once we projectivize them, they form a finite dimension disk. The mapping class $\varphi$ acts continuously on them, which means there is some fixed point, which shall be the measure we assign to $\mu_s$. We thus have that $\varphi(\mu_s) = \lambda'\mu_s$ for some $\lambda'$.

Note that $\mu_s$ and $\mu_u$ have positive self intersection. We thus have the following equality.

$$0 < i(\mu_u, \mu_s) = i(\varphi(\mu_u), \varphi(\mu_s)) = \lambda\lambda' i(\mu_u, \mu_s)$$

This means $\lambda' = \lambda^{-1}$, and that proves the result. □

## References


[Bon01] Francis Bonahon. Geodesic laminations on surfaces. *Contemporary Mathematics*, 269:1–38, 2001.

[Mar16] Bruno Martelli. An introduction to geometric topology. *arXiv preprint arXiv:1610.02592*, 2016.


[‡]One way to see that such a geodesic exists is to use the Anosov closing lemma. But this is probably overkill, and there's surely an easier way of doing this.