Proofs of some technical results in Homotopy Theory

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1 Homotopy Excision Theorem

The homotopy excision theorem is a fairly important and useful result in homotopy theory, and one of its corollaries is the fundamental result in stable homotopy theory.

1.1 Statement and proof of the excision theorem

We'll need a lemma (without proof) and a couple of definitions before we state the theorem.

Lemma 1.1 (Long exact sequence of relative homotopy groups). *For a pair (X, A), the homotopy groups fit into the following long exact sequence.*

$$\cdots \xrightarrow{\partial} \pi_q(A) \xrightarrow{i_*} \pi_q(X) \xrightarrow{j_*} \pi_q(X, A) \xrightarrow{\partial} \pi_{q-1}(A) \xrightarrow{i_*} \cdots$$

The maps i_* and j_* are induced by the inclusions.

Definition 1.1 (n-connectivity for maps). A map f from the pair (X, A) to (Y, B) is said to be n-connected if the induced map on the fundamental group $f_* : \pi_q(X, A) \to \pi_q(Y, B)$ is an isomorphism for $q < n$ and a surjection for $q = n$.

Definition 1.2 (n-connectivity for pairs). A pair (X, A) is said to be n-connected if $i_* : \pi_0(A) \to \pi_0(X)$ is surjective and $\pi_q(X, A) = 0$ for all $q \leq n$. This is equivalent to saying that the inclusion map from $(A, *) \to (X, *)$ is n-connected (this follows from the long exact sequence for relative homotopy groups).

With the terms defined, we can now state the theorem.

Theorem 1.2 (Homotopy excision). *Let X_1 and X_2 be open subspaces of the space X, such that $X = X_1 \cup X_2$, and $X_0 = X_1 \cap X_2$ is not empty. If the pair (X_1, X_0) is m-connected, and (X_2, X_0) is n-connected, then the inclusion map $i : (X_1, X_0) \to (X, X_2)$ is $(m + n)$-connected, for $m \geq 1$ and $n \geq 0$.*

The idea of the proof is borrowed from May [**?**] : we will try to fit in the maps we want to show are isomorphisms, and surjections into a long exact sequence, and try to show that the third term vanishes in an appropriate range of dimensions. For this, we'll need another form of homotopy groups. Put citation

Definition 1.3 (Triad homotopy groups). If X is a space, and X_1 and X_2 are open subspaces, such that the basepoint x_0 lies in $X_0 = X_1 \cap X_2$, then the triad homotopy groups are homotopy classes of maps of tetrads of the following form.

$$\begin{array}{c}(I^q, I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \\ \downarrow \\ (X, X_1, X_2, x_0)\end{array}$$

This is only defined for $q \geq 2$.

Lemma 1.3 (Long exact sequence of triad homotopy groups)**.** *The triad homotopy groups fit into a long exact sequence with relative homotopy groups in the following manner.*

$$\cdots \xrightarrow{\partial} \pi_q(X_1, X_0) \xrightarrow{i_*} \pi_q(X, X_2) \xrightarrow{j_*} \pi_q(X; X_1, X_2) \xrightarrow{\partial} \pi_{q-1}(X_1, X_0) \xrightarrow{i_*} \cdots$$

The maps i_* and j_* are induced by the inclusions, and ∂ is the boundary homomorphism, defined by restricting the map to the face of the cube corresponding to X_1.

The proof that this sequence is exact is similar to the exactness proof for long exact sequence of relative homotopy groups, and hence skipped.

Coming back to the excision theorem, we see that the condition $m \geq 0$ and $n \geq 0$ forces the map at the level of π_0 to be an isomorphism. Furthermore, because $m \geq 1$, by an argument similar to the one in the proof of Seifert–Van Kampen theorem, we get that $\pi_1(X, X_2) = 0$, hence it's an isomorphism. That means we only need to check for $2 \leq q \leq m + n$. By the long exact sequence of triad homotopy groups, it's equivalent to proving the following theorem.

Theorem 1.4. *With the same hypotheses as that of the homotopy excision theorem,*

$$\pi_q(X; X_1, X_2) = 0$$

for all $2 \leq q \leq m + n$.

Proof. It will suffice to prove this result when X is a CW complex, and X_1, X_2, and X_0 are CW subcomplexes. That's because we can approximate a space by a CW complex up to homotopy, and that won't change the homotopy groups. By our hypotheses on the connectivity of (X_1, X_0) and (X_2, X_0), (X_1, X_0) contains no relative q-cells i.e. q-cells outside X_0 for $q \leq m$, otherwise $\pi_q(X_1, X_0)$ wouldn't be 0. Similarly, (X_2, X_0) contains no relative q-cells for $q \leq n$. Furthermore, since we're trying to show a certain map from a compact space is nullhomotopic, it suffices to consider cases where X is a finite CW complex.

We will prove the result by inducting on the number of relative cells of (X_1, X_0) and (X_2, X_0). Since the base case is the hard part, we'll show the induction step first. Suppose we know the result for the triads $(X; X_1', X_2)$ and $(X; X_1, X')$, where X_1 is obtained by attaching one more cell to X_1', and $X' = X_1' \cup_{X_0} X_2$ (The first triad satisfies the induction hypothesis for obvious reasons. The second triad has one relative cell in each component, so that reduces to the base case.).

By the long exact sequence of a triple, we get exactness in the rows of the following diagram.

$$\begin{array}{ccccccc}
\pi_q(X_1', X_0) & \longrightarrow & \pi_q(X_1, X_0), & \longrightarrow & \pi_q(X_1, X_1') & \longrightarrow & \pi_{q-1}(X_1', X_0) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta \\
\pi_q(X', X_2) & \longrightarrow & \pi_q(X, X_2) & \longrightarrow & \pi_q(X, X') & \longrightarrow & \pi_{q-1}(X', X_2)
\end{array}$$

By the induction hypothesis, the maps α and γ are surjective, and δ is injective, which means the map β is surjective. This shows that $\pi_q(X; X_1, X_2) = 0$.

Similarly, if we know the result for $(X; X_1, X_2')$ and $(X; X', X_2)$, where X_2 is obtained by attaching one more cell to X_2', then we have the result for $(X; X_1, X_2)$, since the map $i : (X_1, X_0) \hookrightarrow (X, X_2)$ factors through in the following manner.

$$(X_1, X_0) \overset{i_1}{\hookrightarrow} (X', X_2') \overset{i_2}{\hookrightarrow} (X, X_2)$$

By the induction hypothesis, both i_{1*} and i_{2*} are surjections, hence their composite i_* is a surjection.

All that is left now is to prove the base case, i.e. when both (X_1, X_0) and (X_2, X_0) have one relative cell. Without loss of generality, let $X_1 = X_0 \cup D^k$, where $k > m$ and $X_2 = X_0 \cup D^l$, where $l > n$. We need to show that the associated map of tetrads is nullhomotopic.

$$\begin{array}{c}(I^q, I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \\ \downarrow \\ (X, X_0 \cup D^k, X_0 \cup D^l, x_0)\end{array}$$

Pick interior points $x_1 \in D^k$ and $x_2 \in D^l$. We then have the following maps of triads.

$$(X_1; X_1, X_1 - x_1) \hookrightarrow (X - x_2; X_1, X - \{x_1, x_2\}) \tag{1}$$
$$(X - x_2; X_1, X - \{x_1, x_2\}) \hookrightarrow (X; X_1, X - \{x_1\}) \tag{2}$$
$$(X; X_1, X_2) \hookrightarrow (X; X_1, X - \{x_1\}) \tag{3}$$

The maps labelled 1 and 3 induce isomorphisms at the level of triad homotopy groups. This is easy to see after observing the fact that D^n with an interior point removed can be homotoped to its boundary S^{n-1}. Doing this to the associated disks in map 1 and 3, we see the isomorphisms. Furthermore, $\pi_q(X_1; X_1, X_1 - x_1) = 0$ for $q \geq 2$: this follows trivially from the long exact sequence of triad homotopy groups. That means if we show map 2 induces a surjection at the level of π_*, we'll be done.

Pick any map of tetrads f going into $(X; X_1, X - \{x_1\})$. We need to show f is homotopic to a map into $(X - x_2; X_1, X - \{x_1, x_2\})$ followed by the inclusion map 2. Let $D^k_{1/2}$ and $D^l_{1/2}$ be sub-disks of radius $\frac{1}{2}$. Using the compactness of I^q, it's easy to divide I^q into smaller sub-cubes I^q_α such that if $f(I^q_\alpha)$ intersects $D^k_{1/2}$, it's contained in the interior of D^k and if it intersects $D^l_{1/2}$, it's contained in the interior of D^l. By simplicial approximation, we can make f homotopic (as a map of tetrads) to a map g whose restriction to the $(k-1)$ skeleton of I^q (with its subdivided cubes as cells) (the $(k-1)$ skeleton could possibly be empty as well) does not cover $D^k_{1/2}$, and similarly, whose restriction to $(l-1)$-skeleton does not cover $D^l_{1/2}$. Furthermore, we can make sure that we pick an x_2 in $D^l_{1/2}$ such that $g^{-1}(x_2)$ has dimension at most $q - l$ and it does not lie in the image of the $(l-1)$-skeleton. Although, this is intuitively clear, it requires some work, but that will just obscure the proof, so we leave it be.

Let $\pi : I^q \to I^{q-1}$ be a projection map that discards the last coordinate, and let K be the following space.

$$K = \pi^{-1} \circ \pi \circ g^{-1}(x_2)$$

This means K has dimensions at most 1 more than $g^{-1}(x_2)$. We have the following inequalities.

$$\begin{aligned} \dim K &\leq \dim g^{-1}(x_2) + 1 \\ &\leq q - l + 1 \\ &< m + 1 \end{aligned}$$

The last inequality follows since $l > n$, and $q \leq m + n$. And since $m < k$, and all our dimensions are integers, that lets us directly conclude $\dim K \leq k - 1$. This means $g(K)$ does not cover all of $D^k_{1/2}$. We pick $x_1 \in D^k_{1/2}$ such that it does not lie in $g(K)$. It's not too hard to see that $\pi(g^{-1}(x_1)) \cup \partial I^{q-1}$ and $\pi(g^{-1}(x_2))$ are disjoint (drawing a picture in the case of I^3 helps visualizing the scenario). Both of these are closed subsets of I^{q-1}, hence by Urysohn's lemma, there exists a function $v : I^{q-1} \to I$ such that the following equalities are satisfied. Insert image perhaps?

$$\begin{aligned} v(\pi(g^{-1}(x_1)) \cup \partial I^{q-1}) &= 0 \\ v(\pi(g^{-1}(x_2))) &= 1 \end{aligned}$$

We use this to define a map h from I^{q+1} to I^q.

$$h(r, s, t) = (r, s - s \cdot t \cdot v(r))$$

Let $f' : I^q \to X$ be defined as $f'(r, s) = g \circ h(r, s, 1)$. We have the following observations.

$$\begin{aligned} h(r, s, 0) &= (r, s) \\ h(r, 0, t) &= (r, 0) \\ h(r, s, t) &= (r, s) && \text{if } r \in \partial I^{q-1} \end{aligned}$$

Furthermore, if $h(r, s, t) \in g^{-1}(x_1)$, then $v(r) = 0$, and hence $h(r, s, t) = (r, s)$. Similarly, $h(r, s, t) \in g^{-1}(x_2)$ implies $v(r) = 1$, and $h(r, s, t) = (r, s - st)$. Thus, $g \circ h$ is the required homotopy of tetrads.

□

1.2 Corollaries of the excision theorem

1.2.1 Freudenthal suspension theorem

For any space X, we have the suspension homomorphism, Σ_*, defined as follows.

$$\begin{aligned} \Sigma_q : \pi_q(X) &\to \pi_{q+1}(\Sigma X) \\ \Sigma_q([f] : S^q \to X) &= [f \wedge \mathrm{id}] \end{aligned}$$

Under certain additional conditions on the space X and q, Σ_q is an isomorphism. The excision theorem lets us easily deduce what those conditions are.

Theorem 1.5 (Freudenthal suspension). *If X is n-connected, then the suspension homomorphism is an isomorphism for $q \leq 2n$ and a surjection for $q = 2n + 1$.*

Proof. Let C_+X and C_-X be the upper and lower cones of ΣX. Their intersection is X. The pairs (C_+X, X) and (C_-X, X) are both n-connected (this follows from the fact that cones are contractible, and the long exact sequence of relative homotopy groups). By the homotopy excision theorem, we get that the inclusion of (C_+X, X) into $(\Sigma X, C_-X)$ is an isomorphism until π_{2n} and a surjection for π_{2n+1}. Again, using the long exact sequence of relative homotopy groups, we get that $\pi_*(C_+X, X) \cong \pi_*(X)$, and similarly $\pi_*(\Sigma X, C_-X) \cong \pi_{*+1}(\Sigma X)$. There is a bit of work involved in showing that this isomorphism/surjection is actually the suspension homomorphism, but that's technical, and not too hard. $\square$

1.2.2 Stable homotopy

We now look at the colimit of the following diagram.

$$\pi_q(S^0) \xrightarrow{\Sigma_q} \pi_{q+1}(S^1) \xrightarrow{\Sigma_{q+1}} \pi_{q+2}(S^2) \xrightarrow{\Sigma_{q+2}} \cdots$$

Since S^n is n-connected, after $\frac{q}{2}$ steps, the arrows in the diagram become isomorphisms, which means the colimit of the diagram is what the diagram stabilizes to after $\frac{q}{2}$ steps. We call the colimit the q^{th} stable homotopy group of S^0. From this point on, it's not too hard to show that the stable homotopy functor is a nicely behaved functor: in fact, it's a generalized homology theory.

2 Comparison theorem for cohomology theories

Theorem 2.1 (Comparison Theorem). *If h and k are two reduced cohomology theories satisfying the (DV) axiom, such that there exists a natural transformation χ from h^* to k^* and χ induces a natural isomorphism from $h^n(S^0)$ to $k^n(S^0)$ for all n, then χ is a natural isomorphism of cohomology theories.*

Proof. First, we'll use a simple reduction. For any pointed space X, we'll show if χ is an isomorphism from $h^n(X_+)$ to $k^n(X_+)$, then χ is an isomorphism between $h^n(X)$ and $k^n(X)$. Here, X_+ is the space $X \amalg +$, with the basepoint being $+$. Consider the following cofiber sequence.

$$S^0 \xrightarrow{i} X_+ \to X$$

The map from S^0 to X_+ sends the basepoint of S^0 to $+$, and the other point to the original basepoint of X. This cofiber sequence splits, which means an isomorphism from $h^n(X_+)$ to $k^n(X_+)$ is equivalent to an isomorphism from $h(X)$ to $k(X)$ (by the five lemma).

The next step in the proof is to show the isomorphism for finite CW complexes. This will be done by showing the isomorphism for S^n and D^n for all n, and then using the fact that all finite CW complexes are pushouts of disks and spheres. We have an isomorphism on all spheres from the suspension isomorphism, and that extends to wedges of spheres. And since all disks are contractible, we have an isomorphism there as well. This also tells us that χ is an isomorphism from 0-dimensional CW complexes. We'll proceed by induction at this point. An n skeleton is defined by the following pushout.

$$\begin{array}{ccc} \coprod S^n_+ & \longrightarrow & X^{(n-1)}_+ \\ \downarrow & & \downarrow \\ \coprod D^n_+ & \dashrightarrow & X^{(n)}_+ \end{array}$$

We then apply the Mayer-Vietoris sequence to the subspaces $\coprod D^n_+$ and $X^{(n-1)}_+$. Call these subspaces T_1 and T_2. Their intersection T_0 is the wedge of spheres.

$$\begin{array}{ccccccccc} h^{n+1}(T_1) \oplus h^{n+1}(T_2) & \longleftarrow & h^{n+1}(T_0) & \longleftarrow & h^n(X^{(n)}) & \longleftarrow & h^{n+1}(T_1) \oplus h^{n+1}(T_2) & \longleftarrow & h^{n+1}(T_0) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \chi & & \downarrow \cong & & \downarrow \cong \\ k^{n+1}(T_1) \oplus k^{n+1}(T_2) & \longleftarrow & k^{n+1}(T_0) & \longleftarrow & k^n(X^{(n)}) & \longleftarrow & k^{n+1}(T_1) \oplus k^{n+1}(T_2) & \longleftarrow & k^{n+1}(T_0) \end{array}$$

By the induction hypothesis, all but the middle arrows are isomorphisms, hence by the five lemma, χ is an isomorphism too. This shows χ is a natural isomorphism for finite dimensional CW complexes.

The final step is to show this for infinite dimensional CW complexes. This will require the use of Milnor's theorem B.5. For any infinite dimensional CW complex X, take the filtration consisting of its finite dimensional skeletons. Then we have maps between the following short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {\lim_n}^1 h^{q-1}(X^{(n)}) & \longrightarrow & h^q(X) & \longrightarrow & \lim_n h^q(X^{(n)}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \chi & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & {\lim_n}^1 k^{q-1}(X^{(n)}) & \longrightarrow & k^q(X) & \longrightarrow & \lim_n k^q(X^{(n)}) & \longrightarrow & 0 \end{array}$$

By the proof for the finite dimensional case, we have the all but the middle arrow are isomorphisms. By the five lemma, we get the middle arrow is an isomorphism too, which concludes the proof.

□

3 Brown's representability theorem

In this section, we shall see that all reduced cohomology theories that satisfy the wedge sum (DV) axiom are representable functors, i.e. they are naturally isomorphic to the hom functor in the homotopy category hCW_*. In particular, for a given reduced cohomology theory h^*, we'll construct a sequence of spaces $\mathcal{Z}(n)$, which we'll call a spectrum, such that $\widetilde{h}^n(X)$ is naturally isomorphic to $[X, \mathcal{Z}(n)]$.

3.1 Spectra and cohomology theories

Definition 3.1 (Ω-Spectrum)**.** A spectrum is a $\mathbb{Z}$ indexed sequence of pointed spaces $\mathcal{Z}(n)$ together with structure maps $\sigma_n : \Sigma\mathcal{Z}(n) \to \mathcal{Z}(n+1)$. If the adjoints of the structure maps, i.e. the maps $\widetilde{\sigma}_n : \mathcal{Z}(n) \to \Omega\mathcal{Z}(n+1)$ are homotopy equivalences, then the spectrum is called an Ω-spectrum.

Proposition 3.1. *Given a Ω-spectrum $\mathcal{Z}$, one can define the following functor.*

$$\widetilde{h}^n(X;\mathcal{Z}) = [X, \mathcal{Z}(n)]$$

This is a contravariant functor which satisfies the homotopy invariance (H)*, suspension* (S)*, exactness* (E)*, and the wedge sum* (DV) *axiom. It is therefore a reduced cohomology theory.*

Proof. We'll deal with the axioms one at a time.

Homotopy invariance (H): This is obvious, because we are looking at homotopy classes of maps.

Suspension (S): We need to show there is a natural isomorphism from $\widetilde{h}^n(X)$ to $\widetilde{h}^{n+1}(\Sigma X)$. Note that the adjoint of the structure maps are homotopy equivalences. We therefore have a natural isomorphism.

$$[X, \mathcal{Z}(n)] \cong [X, \Omega\mathcal{Z}(n+1)]$$

On the other hand, since Σ are Ω are adjoints, we have the following natural isomorphism.

$$[X, \Omega\mathcal{Z}(n+1)] \cong [\Sigma X, \mathcal{Z}(n+1)]$$

Composing the two natural isomorphisms, we get our required isomorphism.

Exactness (E): We need to show for any cofibration $i : A \hookrightarrow X$, the following sequence is exact.

$$\widetilde{h}^n(A) \leftarrow \widetilde{h}^n(X) \leftarrow \widetilde{h}^n\,(X/A)$$

Using the cofiber sequence, we get that following sequence is exact.

$$[A, \mathcal{Z}(n)] \leftarrow [X, \mathcal{Z}(n)] \leftarrow [(X/A)\,, \mathcal{Z}(n)]$$

Wedge sum (DV): The functor $[\cdot, \mathcal{Z}(n)]$ satisfies (DV) axiom. This is fairly easy to check. That means $\widetilde{h}^*$ satisfies (DV) axiom.

□

To construct easy examples of spectra, one needs to check that the filtered colimits commute with the loop space functor, at least for nice enough spaces.

3.2 Proof of Brown's representability theorem

Note: This proof is primarily taken from A. J. Tolland's article[**?**].

In the previous section, we saw that if we are given an Ω-spectrum, we can construct a reduced cohomology theory using the spectrum. Brown's representability theorem is the converse of the previous theorem, i.e. given a reduced cohomology theory which satisfies the (DV) axiom, it can be represented by an Ω-spectrum, which is unique up to homotopy. This theorem is fairly technical, and will require the use of the theorem on Milnor exact sequence (theorem B.5).

Theorem 3.2 (Brown's representability theorem)**.** *Let $\widetilde{h}^*$ be a reduced cohomology theory satisfying the (DV) axiom. Then there is an Ω-spectrum $\mathcal{Z}$ such that $\widetilde{h}^n$ is naturally isomorphic to $[\cdot, \mathcal{Z}(n)]$.*

Proof. The proof will have two main parts. The first part will involve constructing the spaces $\mathcal{Z}(n)$ for each n such that there is a natural isomorphism from $\widetilde{h}^n(X)$ to $[X, \mathcal{Z}(n)]$ for all CW complexes X. The second part will involve constructing the structure maps from $\Sigma\mathcal{Z}(n) \to \mathcal{Z}(n+1)$.

Fix an $n \in \mathbb{Z}$. We will construct the space $\mathcal{Z}(n)$ as a CW complex, using finite dimensional skeletons $\mathcal{Z}(n)_k$. For each k, we will also pick a cohomology class $c_n(k)$ in $\widetilde{h}^n(\mathcal{Z}(n)_k)$ such that the map $d_n^m(k) : [S^m, \mathcal{Z}(n)_k] \to \widetilde{h}^n(S^m)$ is an isomorphism for $m < k$ and surjection for $m = k$.

Show that this is a group homomorphism for $m \geq 1$

$$d_n^m(k) : [S^m, \mathcal{Z}(n)_k] \to \widetilde{h}^n(S^m)$$
$$d_n^m(k) : [f] \mapsto f^*(c_n(k))$$

For $k = 0$, we define $\mathcal{Z}(n)_0$ as follows.

$$\mathcal{Z}(n)_0 := \bigvee_{\alpha \in \widetilde{h}^n(S^0)} S^0_\alpha$$

The cohomology group of $\mathcal{Z}(n)_0$ is given by a direct product, since $\widetilde{h}^n$ satisfies the (DV) axiom.

$$\widetilde{h}^n(Z(n)_0) \cong \prod_{\alpha \in \widetilde{h}^n(S^0)} \widetilde{h}^n(S^0_\alpha)$$

Pick the following element as $c_n(0)$.

$$c_n(0) := \prod_{\alpha \in \widetilde{h}^n(S^0)} \alpha$$

Since $k = 0$, we only need to show that $d_n^0(0)$ is a surjection. Pick any $\alpha \in \widetilde{h}^n(S^0)$. Corresponding to this α, there's a copy of S^0 sitting inside $\mathcal{Z}(n)_0$. Let f be the inclusion map of this copy of S^0 into $\mathcal{Z}(n)_0$. Then the induced map on cohomology is the projection map on the α^{th} coordinate, since the cohomology theory satisfies the (DV) axiom. Applying this induced map on $c_n(0)$, we see that in the α^{th} coordinate, it has α, because of the way we defined it. This shows the map is surjective.

To prove the induction step, suppose we have defined the space $\mathcal{Z}(n)_k$ and $c_n(k)$ that satisfy the required properties. Let $K_k \trianglelefteq \left[S^k, \mathcal{Z}(n)_k\right]$ be the kernel of the map $d_n^k(k)$. We construct the following map.

The cofibration probably works if you take a subset of K_k that does not contain the constant map

$$\phi_n(k) : \bigvee_{x \in K_k} S^k \to \mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}$$

This map is obtained by taking the wedge of maps from S^k to $\mathcal{Z}(n)_k$ which are contained in K_k. This is a cofibration . By the (DV) axiom, we have the following cohomology groups.

Not sure how to show this, or whether this is entirely correct. Need to check later

$$\widetilde{h}^n\left(\mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}\right) = \widetilde{h}^n(\mathcal{Z}(n)_k) \times \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1})$$

From this, we can immediately see that the elements of $\widetilde{h}^n\left(\mathcal{Z}(n)_k \vee \bigvee_{y \in \widetilde{h}^n(S^{k+1})} S^{k+1}\right)$ of the form $(c_n(k), \bullet)$ (where $\bullet$ is any arbitrary element) is in the kernel of $\phi_n^*(k)$. Define $\mathcal{Z}(n)_{k+1}$ to be the cofiber of the map $\phi_n(k)$, and let the map to the cofiber be $b_n(k)$. By the exactness axiom, we have that the following sequence is exact.

$$\widetilde{h}^n(\mathcal{Z}(n)_{k+1}) \xrightarrow{b_n^*(k)} \widetilde{h}^n(\mathcal{Z}(n)_k) \times \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1}) \xrightarrow{\phi_n^*(k)} \prod_{x \in K_k} \widetilde{h}^n(S^k)$$

Pick the following element $A \in \prod_{y \in \widetilde{h}^n(S^{k+1})} \widetilde{h}^n(S^{k+1})$.

$$A := \prod_{\alpha \in \widetilde{h}^n(S^{k+1})} \alpha$$

The element $(c_n(k), A)$ lies in the kernel of $\phi_n^*(k)$, which means it lies in the image of $b_n^*(k)$. We define $c_n(k+1)$ to be a pre-image of $(c_n(k), A)$. Seeing that the associated map $d_n^m(k+1)$ is surjective for $m = k+1$ is easy enough. The proof is the same as that in the case of $d_n^0(0)$.The trickier part is showing injectivity for $m < k+1$. Since $d_m^n(k)$ is a group homomorphism (for $M \geq 1$), for $m \geq 1$, it will suffice to show the kernel is trivial. Pick an element, say $[f]$ in the kernel. We need to show that f is a nullhomotopic map. But at each step, we coned off the kernel of $d_n^m(m)$. That means f is nullhomotopic. This shows the injectivity and hence the isomorphism for $m < k+1$.

Maybe write the proof anyways. See if it adds to the clarity at all

I'm skipping the proof of the case when $m = 0$. I think it should be fairly easy once I figure out why the map is a cofibration.

Next, we define $\mathcal{Z}(n)$ the colimit of the following diagram.

$$\mathcal{Z}(n)_0 \overset{b_n(0)}{\hookrightarrow} \mathcal{Z}(n)_1 \overset{b_n(1)}{\hookrightarrow} \mathcal{Z}(n)_2 \overset{b_n(2)}{\hookrightarrow} \cdots$$

Note that $\mathcal{Z}(n)_k$ are CW subcomplexes of $\mathcal{Z}(n)$, in particular, we can appeal to Milnor's theorem B.5, i.e. the following sequence is exact.

$$0 \to \lim_k{}^1 \widetilde{h}^{n-1}(\mathcal{Z}(n)_k) \to \widetilde{h}^n(\mathcal{Z}(n)) \to \lim_k \widetilde{h}^n(\mathcal{Z}(n)_k) \to 0$$

Furthermore the element $(c(n)_0, c(n)_1, c(n)_2, \ldots)$ lies in $\lim_k \widetilde{h}^n(\mathcal{Z}(n)_k)$ since we pick $c(n)_{k+1}$ as a preimage of $c(n)_k$. By exactness, we get a preimage c_n in $\widetilde{h}^n(\mathcal{Z}(n))$. We define a map

$d_n^m : [S^m, \mathcal{Z}(n)] \to \widetilde{h}^n(S^m)$ which sends $[f]$ to $f^*(c_n)$. Because of the inductive construction, we know this is an isomorphism for all $m \geq 0$ (To see this, observe that a map from a compact space like S^m factors through a finite stage in the colimit $\mathcal{Z}(n)$). This can be extended to a natural isomorphism for all finite CW complexes. This can be done by applying Mayer-Vietoris to the $n-1$ skeleton and the discs being attached and then applying the five lemma, like we did in section 2. Now that we know how to represent the individual functors $\widetilde{h}^n$, we need to construct the structure maps from the suspension homomorphism of the cohomology theory. Let T_n be the suspension homomorphism from $\widetilde{h}^n(X)$ to $\widetilde{h}^{n+1}(\Sigma X)$. If we set X to be $\mathcal{Z}(n)$, we have the following homomorphism.

$$T_n : \widetilde{h}^n(\mathcal{Z}(n)) \to \widetilde{h}^{n+1}(\Sigma \mathcal{Z}(n+1))$$

But this is equivalent to the following homomorphism.

$$\widetilde{T}_n : [\mathcal{Z}(n), \mathcal{Z}(n)] \to [\Sigma \mathcal{Z}(n), \mathcal{Z}(n+1)]$$

We do the most obvious thing, i.e. apply $\widetilde{T}_n$ to the homotopy class of the identity map, and we pick a map to be our structure map from the resulting homotopy class. Show that this forms an Ω-spectrum

□

This result enables us to study any reduced cohomology theory by studying its associated spectrum. This lets us study many cohomology theories that were intractable by the usual methods, e.g. cobordism, which is represented by the Thom spectrum.

A Definitions and notation

Definition A.1 (Suspension of a pointed space)**.** The suspension ΣX of a pointed space X is the smash product $S^1 \wedge X$.

Definition A.2 (Loop space of a pointed space)**.** The loop spaces ΩX of a pointed space X is the set of all pointed maps from S^1 to X with the compact-open topology.

Definition A.3 ($\lim^1$)**.** Let T be the category of towers of abelian groups, i.e. $\mathbb{N}$ indexed set of abelian groups G_i with maps $f_i : G_i \to G_{i-1}$, and maps are set of arrows that make the whole thing commute . Then $\lim$ is a left exact functor from T to AbGrp, and we define $\lim^1$ to be the first right derived functor of $\lim$.

Check that this category has enough injectives

B Some useful lemmas and theorems

Note: Although we state many of the lemmas here for TOP, they are also true for TOP_*, and the proof is similar.

Lemma B.1. *If $i : A \hookrightarrow X$ is a cofibration (in the category TOP), then the mapping cone $C(i)$ is homotopy equivalent to X/A.*

Proof. We will first construct the maps to and from $C(i)$ to X/A. The maps from $C(i)$ to X/A is the map that collapses the cone of A to a point corresponding to A in X/A. Now consider a map from $H : A \times I$ to $C(i)$, such that H contracts A to a point in $C(i)$, starting from the inclusion of A in X. Let the map j from X to $C(i)$ be the inclusion map. Since i is a cofibration, we can extend H with the initial condition j to a map $J : X \times I \to C(i)$. But $J(\cdot, 1)$ collapses A to a point. That means it factors through a X/A. This gives us a map k from X/A to $C(i)$.

The fact that these maps are homotopy inverses can be verified using the homotopy J. □

Not sure of this. Verify later.

Lemma B.2. *In the category TOP, the following sequence is h-coexact.*

$$A \xrightarrow{f} B \xrightarrow{i} C(f)$$

That means for any space Z, the following sequence of abelian groups is exact.

$$[A, Z] \leftarrow [B, Z] \leftarrow [C(f), Z]$$

Proof. If an element $[c] \in [B, Z]$ goes to 0 in $[A, Z]$, that means $c \circ f : A \to Z$ is nullhomotopic, where c is a representative of $[c]$. But that means there is some function $d \in C(f)$ such that $c = d \circ i$. This shows the exactness of the sequence. □

Lemma B.3. *If K is a compact space, let A_i be a sequence of spaces where points are closed, and A is the colimit of the following diagram:*

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots$$

where all the embeddings are closed, then a map from K to A factors finitely through some A_i.

Proof. Let $J = f(K)$ be the compact image of K in A. For each set $A_i \setminus A_{i-1}$, pick an element c_i of J in the set, if J intersects $A_i \setminus A_{i-1}$. Since A_i's are closed, that means the subset c_i has the discrete topology. Furthermore, since points are closed, the set $\{\cup c_i\}$ is a closed subset of J, hence compact. And compact spaces with discrete topology are finite. That means only finitely many $A_i \setminus A_{i-1}$ intersect J. This means the map factors through at some finite stage. □

Theorem B.4 (Alternative characterization of $\lim^1$). *If F is an object in the tower category, then $\lim^1(F)$ is the cokernel of the following map.*

$$\alpha_F : \prod_{i \in \mathbb{N}} F_i \to \prod_{i \in \mathbb{N}} F_i$$
$$\alpha_F : (g_0, g_1, g_2, \ldots) \mapsto (g_0 - f_1(g_1), g_1 - f_2(g_2), \ldots)$$

Proof. The first step in characterizing $\lim^1$ in the following manner is to pick an appropriate injective resolution. Let F be a tower of abelian groups and I an injective tower it maps into via a monomorphism m.

$$\begin{array}{ccccccccc} F_0 & \xleftarrow{f_1} & F_1 & \xleftarrow{f_2} & F_2 & \xleftarrow{f_3} & F_3 & \xleftarrow{f_4} & \cdots \\ \downarrow{\scriptstyle m_0} & & \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_2} & & \downarrow{\scriptstyle m_3} & & \\ I_0 & \xleftarrow{i_1} & I_1 & \xleftarrow{i_2} & I_2 & \xleftarrow{i_3} & I_3 & \xleftarrow{i_4} & \cdots \end{array}$$

Without losing any generality, we can assume all the maps i_k in I are surjective. Otherwise, we just replace I_k by $\bigoplus_{j=0}^{k} I_j$, and have the maps on all but the last coordinate be the identity. This is important, because we'll need surjectivity of the maps later. We can now construct an injective resolution of F as the following exact sequence.

$$0 \to F \xrightarrow{m} I \xrightarrow{q} \operatorname{coker}(m) \to 0$$

The first derived functor is the homology at $\lim(\operatorname{coker}(m))$ of the following sequence.

$$0 \to \lim(F) \xrightarrow{\lim(m)} \lim(I) \xrightarrow{\lim(q)} \lim(\operatorname{coker}(m)) \to 0$$

Now, just like α_F was defined in the statement of the theorem, we define α_I and $\alpha_{\operatorname{coker}(m)}$. Then we get the following short exact sequence of chain complexes (the rows are exact).

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_i F_i & \xrightarrow{m} & \prod_i I_i & \xrightarrow{q} & \prod_i \operatorname{coker}(m) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_F} & & \downarrow{\scriptstyle \alpha_I} & & \downarrow{\scriptstyle \alpha_{\operatorname{coker}(m)}} & & \\ 0 & \longrightarrow & \prod_i F_i & \xrightarrow{m} & \prod_i I_i & \xrightarrow{q} & \prod_i \operatorname{coker}(m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We can apply the snake lemma to get the following long exact sequence.

$$0 \longrightarrow \ker(\alpha_F) \xrightarrow{m_*} \ker(\alpha_I) \xrightarrow{q_*} \ker(\alpha_{\mathrm{coker}(m)}) \xrightarrow{\partial} \mathrm{coker}(\alpha_F) \xrightarrow{m^*} \mathrm{coker}(\alpha_I) \xrightarrow{q^*} \mathrm{coker}(\alpha_{\mathrm{coker}(m)}) \longrightarrow 0$$

But we see from the definition of $\lim$ that the kernels of α are precisely the $\lim$. Thus we have the following long exact sequence.

$$0 \longrightarrow \lim(F) \xrightarrow{\lim(m)} \lim(I) \xrightarrow{\lim(q)} \lim(\mathrm{coker}(m)) \xrightarrow{\partial} \mathrm{coker}(\alpha_F) \xrightarrow{m^*} \mathrm{coker}(\alpha_I) \xrightarrow{q^*} \mathrm{coker}(\alpha_{\mathrm{coker}(m)}) \longrightarrow 0$$

The last step in the proof will be to show that $\mathrm{coker}(\alpha_I)$ is 0, in which case $\lim^1(F)$ is isomorphic to $\mathrm{coker}(\alpha_F)$. Showing that $\mathrm{coker}(\alpha_I)$ is 0 is equivalent to showing that α_I is surjective. To see this, pick any element $(j_0, j_1, j_2, \ldots) \in \prod_i I_i$. We need to find an element $(k_0, k_1, k_2, k_3, \ldots)$ such that we have the following equalities.

$$\begin{aligned} j_0 &= k_0 - i_1(k_1) \\ j_1 &= k_1 - i_2(k_2) \\ j_2 &= k_2 - i_3(k_3) \\ &\vdots \end{aligned}$$

But notice that we constructed I such that all the i_k are surjective. That means this system of equations can be solved simultaneously and $\mathrm{coker}(\alpha_I)$ is 0. This shows the result.

□

Theorem B.5 (Milnor exact sequence). *If $\{i_n : X_n \hookrightarrow X_{n+1}\}$ for $n \in \mathbb{N}$ are a sequence of nested (pointed) CW subcomplexes such that $X = \bigcup_n X_n$, and $\widetilde{h}^*$ is a reduced cohomology theory, then we have the following exact sequence for all $i \geq 1$.*

$$0 \to \lim_n{}^1 \widetilde{h}^{i-1}(X_n) \to \widetilde{h}^i(X) \to \lim_n \widetilde{h}^i(X_n) \to 0$$

Proof. We first construct the mapping telescope of the inclusions. We start with the (unpointed) space $X \times \mathbb{R}_+$. We then consider the following subspace S.

$$S = \bigcup_{i \in \mathbb{N}} X_i \times [i, i+1]$$

We then quotient S by the subspace $* \times \mathbb{R}_+$ and we use the quotient out space as the basepoint of the pointed mapping telescope T.

We define two subspaces of the mapping telescope; we'll apply Mayer-Vietoris to these subspaces.

$$T_1 := \bigcup_{i \in \mathbb{N}} X_{2i+1} \times [2i+1, 2i+2]$$
$$T_2 := \bigcup_{i \in \mathbb{N}} X_{2i} \times [2i, 2i+1]$$

To be more precise, we should be taking open neighbourhoods of these spaces to apply Mayer-Vietoris, but the open neighbourhoods deformation retract to these spaces anyways, so it's alright. Observe that the intersection T_0 of T_1 and T_2 is homeomorphic to the wedge of all the X_i. By similar reasoning (although we replace homeomorphism by homotopy equivalence), we get that T_1 is homotopy equivalent to the wedge of all the X_i for odd i and T_2 the wedge of X_i for all even i. We consider the Mayer-Vietoris sequence, and use the (DV) axiom to get isomorphisms.

$$
\begin{array}{ccccccc}
\widetilde{h}^{q+1}(T) & \longleftarrow & \widetilde{h}^q(T_0) & \xleftarrow{\phi_q} & \widetilde{h}^q(T_1) \oplus \widetilde{h}^q(T_2) & \longleftarrow & \widetilde{h}^{q+1}(T) \\
 & & \downarrow \cong & & \downarrow \cong & & \\
 & & \prod_{i\in\mathbb{N}} \widetilde{h}^q(X_i) & & \prod_{i\in\mathbb{N}} \widetilde{h}^q(X_{2i}) \oplus \prod_{i\in\mathbb{N}} \widetilde{h}^q(X_{2i+1}) & & \\
 & \swarrow \cdot(-1)^i & & \swarrow m_q & & & \\
\prod_{i\in\mathbb{N}} \widetilde{h}^q(X_i) & & & & & &
\end{array}
$$

We need to determine how the map m_q should be defined to make the diagram commute. We do this by seeing how the map behaves on the basis. First, we take $x_0 \in \widetilde{h}^q(X_0)$. The first isomorphism takes it to $(x, 0)$, the map ϕ_q takes it $x - 0$ and finally it goes to x. Next, we see how $x_1 \in \widetilde{h}^q(X_1)$ behaves. It first gets sent to $(i_0^* x_1, x_1)$ and then to $i_0^* x_1 - x_1$. Finally, the $(-1)^i$ sends it to $x_1 - i_0^* x_1$. Similarly, $x_2 \in \widetilde{h}^q(X_2)$ gets sent to $x_2 - i_1^* x_2$, and so go on the rest of the basis elements. We now have the following long exact sequence.

$$\widetilde{h}^{q+1}(T) \longleftarrow \prod_{i\in\mathbb{N}} \widetilde{h}^q(X_i) \xleftarrow{m_q} \prod_{i\in\mathbb{N}} \widetilde{h}^q(X_i) \longleftarrow \widetilde{h}^q(T)$$

From this, we get the following short exact sequence.

$$0 \longleftarrow \ker(m_q) \longleftarrow \widetilde{h}^q(T) \longleftarrow \operatorname{coker}(m_{q-1}) \longleftarrow 0$$

From theorem B.4, we see that $\ker(m_q)$ is $\lim_n(\widetilde{h}^q(X_n))$ and $\operatorname{coker}(m_{q-1})$ is ${\lim_n}^1(\widetilde{h}^{q-1}(X_n))$. This proves the theorem.

$\square$