The Siegel-Veech transform

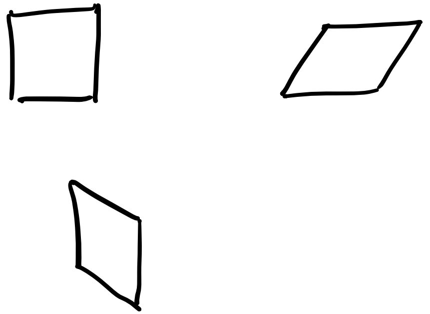
Outline

1. What is a translation surface?
2. What can we count?
3. The Siegel-Veech transform
4. Proving the counting result using equidistribution

1. What is a translation surface?

A translation surface is a finite collection of polygons in $\mathbb{C}$ along with some gluing data, up to an equivalence relation.

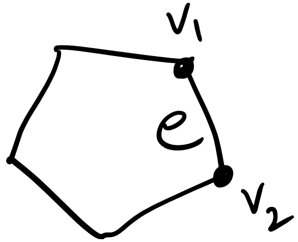
Running example



Gluing data

Running example

To each edge e, associate a complex number $h(e)$.



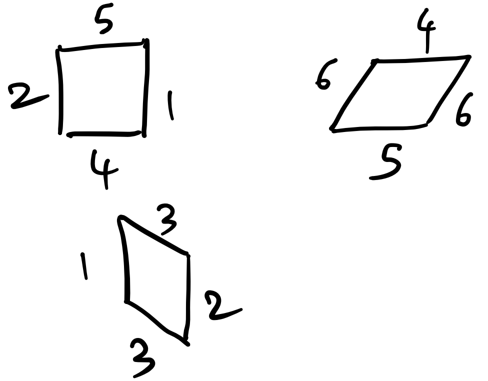
$h(e) = v_1 - v_2$

Gluing rules

i) Every edge is glued to exactly one other edge

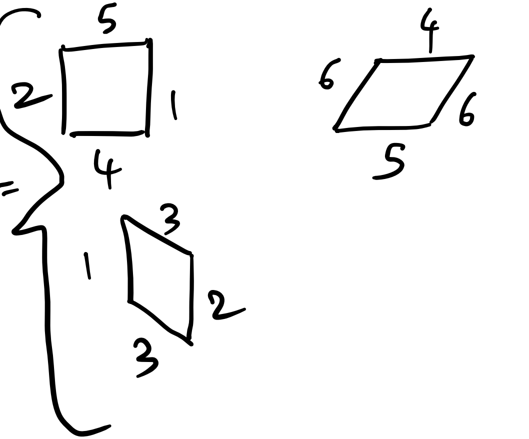
ii) Edges e_1 & e_2 can be glued only if

$h(e_1) = -h(e_2)$



Equivalence relation

$T_1 =$



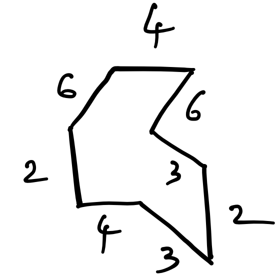
Two translation surfaces are equivalent if there is a bijective map σ

$$\sigma : T_1 \setminus FS_1 \longrightarrow T_2 \setminus FS_2$$

FS_1: Finite set containing vertices

FS_2: Finite set containing vertices

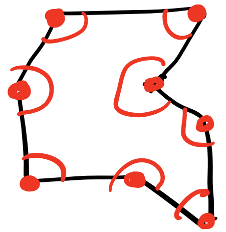
$T_2 =$



σ, when considered in local $\mathbb{C}$ charts, is just a translation.

Objects on a translation surface

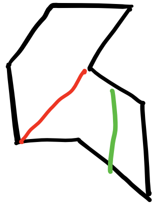
i) Singularities : Points around which angle $= 2n\pi$, $n \geq 2$



Angle of 6π around singularity

Wanna count these

ii) Cylinder vectors : Closed straight lines not passing through singularities



iii) Saddle connection : Straight line passing between a pair of singularities

Can associate a pair of vectors in $\mathbb{R}^2$ to each CV or SC.

The moduli space $\mathcal{M}$

If we fix the degree of each singularity (represented by $\mathcal{K}$)

the set of all translation surfaces with singularity set $\mathcal{K}$ forms a manifold.

We denote this manifold $\mathcal{M}$.

$SL(2,\mathbb{R})$ action on $\mathcal{M}$

We have a linear action of $SL(2,\mathbb{R})$ on $\mathbb{C}$.

$\downarrow$

Action on polygons while respecting gluing rule & equivalence relation

$\downarrow$

Action of $SL(2,\mathbb{R})$ on $\mathcal{M}$

We have a Lebesgue class measure on $\mathcal{M}$. (Masur-Veech)

Thm) μ_{MV} is $SL(2,\mathbb{R})$ invariant & ergodic.

Furthermore, we have a "classification" of $SL(2,\mathbb{R})$ ergodic measures.

Thm (EMM) | $SL(2,\mathbb{R})$ ergodic measures are the canonical Lebesgue class measures on AIS.

2. What can we count?

For each SC/CV, we have a well defined pair of vectors in $\mathbb{R}^2$.

For translation surface S

$V_{cv}(S)$ = Multiset of vectors associated to all cylinder vectors in S

$V_{sc}(S)$ = Same as above, but with saddle connections

$$\#\left(V_{cv}(S) \cap B(0,R)\right)$$

$=$ Cylinder vectors of length $\leq R$

$$\#\left(V_{sc}(S) \cap B(0,R)\right)$$

$=$ Saddle connections of length $\leq R$

The main counting theorem

Let $V : (\mathcal{M}, \mu) \longrightarrow$ Multisets in $\mathbb{R}^2 \setminus \{0,0\}$

satisfying the following conditions

(A) For all $g \in SL(2,\mathbb{R})$

$$V(gS) = gV(S)$$

(B) For any $S \in \mathcal{M}$, $\exists\ c(S) > 0$ s.t

$$\#(V(S) \cap B(0,R)) \leq c(S) \cdot R^2$$

$c(S)$ can be chosen uniformly on compact sets.

(C_μ) For large enough R & small enough ε, $\#(V(S) \cap B(0,R))$ is in $L^{1+\varepsilon}(\mu)$

then $\forall\ \mu$ a.e S

$$\lim_{R\to\infty} \frac{\#(V(S) \cap B(0,R))}{\pi R^2} = C_{V,\mu}$$

The main counting theorem

(A) For all $g \in SL(2,\mathbb{R})$

$$V(gS) = gV(S)$$

(B) For any $S \in \mathcal{M}$, $\exists\ c(S) > 0$ s.t

$$\#(V(S) \cap B(0,R)) \leq c(S) \cdot R^2$$

$c(S)$ can be chosen uniformly on compact sets.

(C_μ) For large enough R & small enough ε, $\#(V(S) \cap B(0,R))$ is in $L^{1+\varepsilon}(\mu)$

- Hypothesis (A) is fairly easy to check.
- (B) & (C_μ) involve doing some actual geometry.
- C_μ also depends upon the choice of the ergodic measure.

Other settings

The statement is quite general. (i.e can replace (M, μ) with any ergodic setup & $\mathbb{Z}$ with m).

$$SL(m, \mathbb{R}) \circlearrowright \left(\frac{SL(m, \mathbb{R})}{SL(m, \mathbb{Z})}, \text{Haar} \right)$$

V is set of vectors in corresponding lattice (or primitive vectors)

3. The Siegel-Veech transform

Suppose $f \in C_0^\infty(\mathbb{R}^2)$ ← Need C^∞ for technical reason

Example one ought to think of: smoothened out indicator of ball or annulus

$$\hat{f} : \mathcal{M} \longrightarrow \mathbb{R}$$

$$\hat{f}(S) := \sum_{v \in V(S)} f(v)$$

Assumption (B) guarantees well definedness & uniform bd on compacts.

(C_μ) guarantees $\hat{f} \in L^{1+\varepsilon}(\mathcal{M}, \mu)$

Siegel-Veech constants

Thm | Let μ be an $SL(2,\mathbb{R})$ ergodic measure, & V be a function satisfying (A), (B), & (C_μ).

Then

$$\int_{\mathcal{M}} \hat{f}(S)\, d\mu(S) = c_{V,\mu} \int_{\mathbb{R}^2} f(x,y)\, dx\, dy.$$

$c_{V,\mu} = 0$

$\Updownarrow$

$\hat{f} = 0 \quad \mu\text{.a.e}$

Compare this to the other counting result when $f = \chi_{B(0,R)}$.

Emphasize what this result says about the AVERAGE translation surface

Proof

Consider the linear functional ϕ on $C_0^\infty(\mathbb{R}^2)$.

$$\phi(f) := \int_M \hat{f}(s) \, d\mu$$

We have an $SL(2,\mathbb{R})$ action on $C_0^\infty(\mathbb{R}^2)$

$$(\gamma \cdot f)(x) = f(\gamma x).$$

But μ is also $SL(2,\mathbb{R})$ invariant.

Let's compute

$$\phi(\gamma \cdot f)$$

Proof (contd.)

$$\phi(\gamma\cdot f) := \int_M \widehat{\gamma\cdot f}(S)\, d\mu = \int_M \sum_{v\in V(S)} \gamma\cdot f(v)\, d\mu$$

$$= \int_M \sum_{v\in V(S)} f(\gamma v)\, d\mu = \int_M \sum_{w\in V(\gamma S)} f(w)\, d\mu$$

$$= \int_M \hat{f}(\gamma S)\, d\mu = \int_M \hat{f}(S)\, d\mu$$

ϕ is an $SL(2,\mathbb{R})$ invariant functional.

Proof (contd.)

Recall that $SL(2,\mathbb{R})$ action on $\mathbb{R}^2$ has two orbits: 0 & $\mathbb{R}^2 \setminus \{0\}$.

$$\phi(f) = a \cdot f(0,0) + b \int_{\mathbb{R}^2} f(x,y)\, dx\, dy$$

Need to show $a = 0$.

Proof (contd.)

Define a sequence of approximating functions f_n.

$$f_n(z) = \begin{cases} 0 & \text{if } |z| < \frac{1}{n} \\ f(z) & \text{if } |z| > \frac{2}{n} \\ \text{smoothly interpolates in b/w.} \end{cases}$$

Idea: Show $\lim_{n \to \infty} \phi(f_n) = \phi(f)$.

Since $f_n(0) = 0 \ \forall n$, this will imply $a = 0$

Proof (contd.)

$$\phi(f_n) = \int_{\mathcal{M}} \sum_{v \in V(s)} f_n(v) \, d\mu$$

Since $f_n \to f$ p.w on $\mathbb{R}^2 \setminus 0$, dominated convergence gives us the result.

The constant b is $c_{v,\mu}$.

For $c_{v,\mu}$ to be 0, ϕ has to be 0.

That'll happen if $\hat{f} = 0$ μ a.e. $\square$

4. Counting from equidistribution

Equidistributing wavefront points

We'll prove the counting result for a special class of translation surfaces.

$$\lim_{t \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(a_t r_\theta S)\, d\theta = \int_{\mathcal{M}} \hat{f}(R)\, d\mu(R) \qquad (*)$$

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$a_t r_\theta S$ is like a wavefront emanating from S.

Counting result for equidistributing wavefronts

Thm If $S \in \mathcal{M}$ satisfies $(*)$, then

$$\lim_{R \to \infty} \frac{\#\left(V(S) \cap B(0,R)\right)}{R^2} = \pi \, c_{V,\mu}$$

Wavefront average estimates on $\mathbb{R}^2$

To estimate

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{f}(a_t r_\theta S)\, d\theta \quad , \quad \text{we will need to}$$

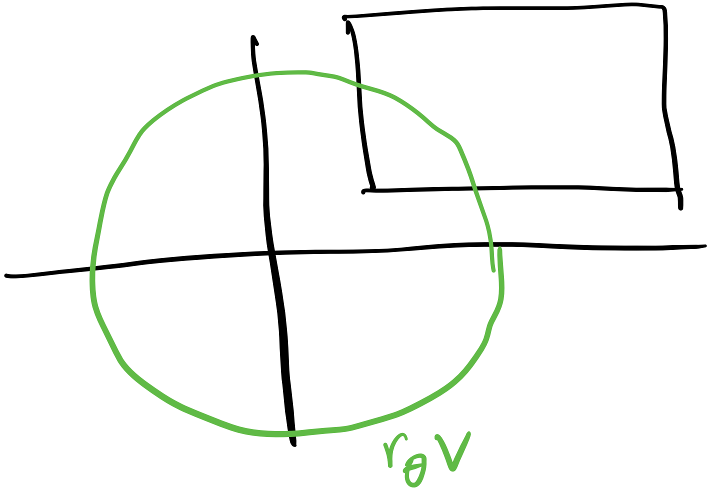
$$\text{estimate} \quad \frac{1}{2\pi}\int_0^{2\pi} f(a_t r_\theta v) \quad \text{for } v \in \mathbb{R}^2$$

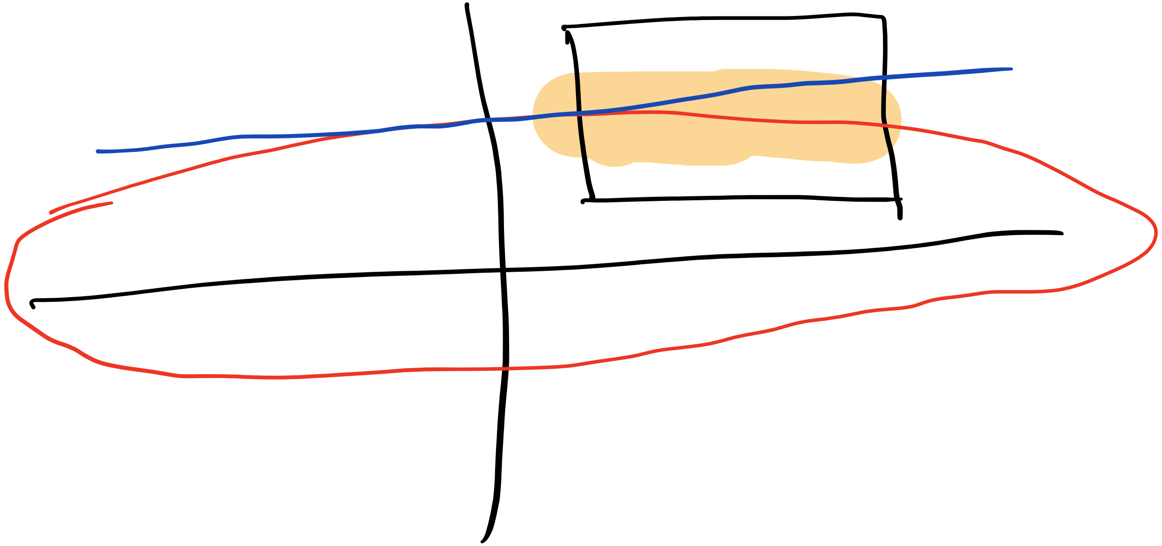
because

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{f}(a_t r_\theta S) = \sum_{v \in V(S)} \frac{1}{2\pi}\int_0^{2\pi} f(a_t r_\theta v)$$

Wavefront average estimates on $\mathbb{R}^2$

WLOG we can assume f is the indicator of some rectangle





Wavefront average estimates on $\mathbb{R}^2$

Lemma

$$\left| \frac{e^{2t}}{2\pi} \int_0^{2\pi} f(a_t r_\theta v)\, d\theta - J_f\left(\|v\| e^{-t}\right) \right| < \varepsilon$$

$$J_f(y) = \frac{1}{2\pi y} \int_{-\infty}^{\infty} f(x, y)\, dx \quad dy$$

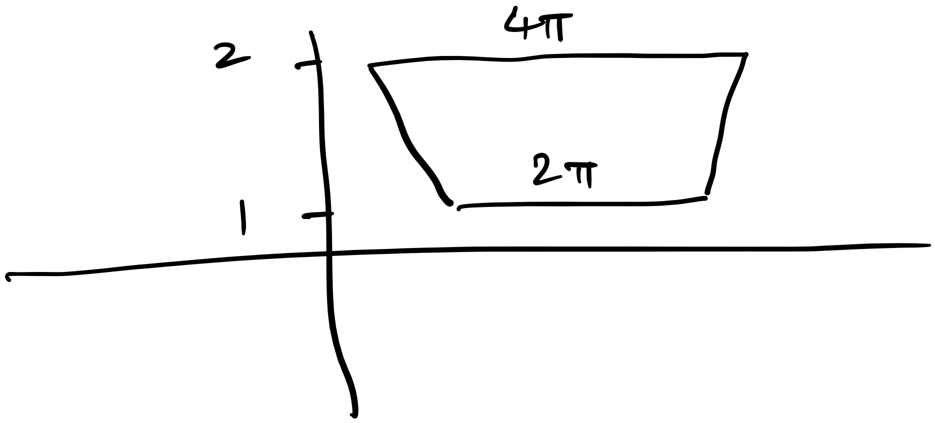
for large enough $\|v\|$ & t

Proof in special case

Let χ be the indicator of $[1,2]$.

Then $\chi(\|v\| e^{-t})$ is 1 for $v \in$ Annulus $(e^t, 2e^t)$

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the indicator of



$J_f = \chi$

For large enough $\|v\|$ & t, we get

$$\frac{e^{2t}}{2\pi}\int_0^{2\pi} f(a_t r_\theta v)\,d\theta - \varepsilon$$

$$\geq$$

$$\chi(\|v\| e^{-t})$$

$$\geq$$

$$\frac{e^{2t}}{2\pi}\int_0^{2\pi} f(a_t r_\theta v)\,d\theta + \varepsilon$$

Sum both sides over all $v \in V(S)$

$$\sum_{v \in V(S)} \int_0^{2\pi} f(a_t r_\theta v)\, d\theta \pm \varepsilon$$

$$= \frac{e^{2t}}{2\pi} \int_0^{2\pi} \hat{f}(a_t r_\theta S) \pm c e^{2t} \cdot \varepsilon$$

$$\sum_{v \in V(S)} \chi(\|v\| e^{-t}) = \#\left(V(S) \cap \text{Annulus}(e^t, 2e^t) \right)$$

Divide by e^{2t}

$$\frac{1}{2\pi}\int_0^{2\pi} \hat{f}(a_t r_\theta S)\, d\theta - C\varepsilon$$

$$\leq \frac{\#\left(V(S) \cap \operatorname{Ann}(e^t, 2e^t)\right)}{e^{2t}}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(a_t r_\theta S)\, d\theta + C\varepsilon$$

As t gets larger, we can make ε go to 0 & the wavefront average becomes μ average

$$\lim_{t\to\infty} \frac{\#\left(V(S) \cap \mathrm{Ann}(e^t, 2e^t)\right)}{e^{2t}}$$

$$= \int_{\mathcal{M}} \hat{f}(S)\, d\mu$$

$\parallel$ Siegel - Veech formula

$$c_{V,\mu} \int_{\mathbb{R}^2} f = (3\pi)\, c_{V,\mu} \qquad \square$$

Reducing to the special case

We've shown the counting result for translation surfaces S from which wavefronts equidistribute.

A theorem of Nevo says that a.e. point S does the job.

Thm) (Nevo) For a.e $S \in (\mathcal{M}, \mu)$ & $f \in L^{1+\varepsilon}$

$$\lim_{t \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta S)\, d\theta = \int_{\mathcal{M}} f(R)\, d\mu(R)$$

This proves the main theorem

Things that were swept under the rug.

1) Wavefront averages are convolved with a probability density φ centered at 0.

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} \varphi(\tau - t) \frac{1}{2\pi} \int_0^{2\pi} f(a_t r_\theta S)\, d\theta$$

2) Same for SV transforms of indicator functions.

$SL(2,\mathbb{R})$ ergodic measures on translation surfaces are classified.