THE MOMENT MAP FOR SYMPLECTIC TORIC VARIETIES

SAYANTAN KHAN

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The standard way of defining a toric variety is that it's a variety which contains a dense open subset isomorphic to the algebraic torus $(\mathbb{C}^*)^n$, which acts on the variety in a nice manner, i.e. the restriction of the action to the torus is left multiplication. Although this definition is technically correct and concise, it doesn't fully elucidate why toric varieties form an important class of examples. They are important because they can be constructed via purely combinatorial data: a toric variety can be completely specified by a convex lattice polytope in $\mathbb{R}^n$. This has many ramifications, one being that one can construct a large class of examples of varieties to test conjectures on, and another being that the geometry of the toric variety translates to the geometry of the associated polytope. This makes performing computations on toric varieties feasible. Of course, in practice this means that one must have a way of constructing the associated polytope from a given toric variety. We'll see how this is done, and to do this, we'll borrow a tool from symplectic geometry called the moment map. This will also illustrate the interplay between toric varieties and symplectic geometry.

1. Toric Varieties

Recall that in the introduction we defined toric varieties as varieties with an open dense torus subset acting on it, and hinted at a more combinatorial construction of toric varieties. We'll perform this construction in two steps. We'll first discuss how to get toric varieties from combinatorial objects called fans, and then discuss how to get fans from lattice polytopes in $\mathbb{R}^n$. The basic building blocks of fans are cones, which are given by the following definition.

Definition 1.1. A cone C in $\mathbb{R}^n$ generated by $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is the following set.

$$C = \left\{ \sum_i a_i v_i \mid a_i \geq 0 \right\}$$

The cone is called rational if the generators $\{v_1, \ldots, v_k\}$ are rational, and smooth[1] is the generators generate the lattice $\mathbb{Z}^n$ as well.

We also need the notion of a face of a cone.

Definition 1.2. A face of a cone C is the intersection of C with the kernel of a linear functional f on $\mathbb{R}^n$.

Note that this definition implies C is a face of itself, and also that any face of a cone is also a cone. Now that we have defined a cone and a face, we can define what a fan is. Informally, fans are to cones what simplicial complexes are to simplices.

Definition 1.3. A fan in $\mathbb{R}^n$ is a collection $\mathcal{U}$ of cones in $\mathbb{R}^n$, which satisfy the following properties.

- If C is a cone in $\mathcal{U}$, every face of C is also in $\mathcal{U}$.
- If C and C' are cones in $\mathcal{U}$, then their intersection is a face of both of them.

[1] The term smooth comes from the fact that the associated toric variety is smooth iff the generators form a basis of $\mathbb{Z}^n$.

Although it isn't strictly standard, we'll also assume that all our fans are top dimensional, i.e. the dimension of the largest cone in the fan is n, where n is the dimension of the ambient space.

To construct toric varieties from a fan, start with a cone C, and then look at the dual cone C^*. This is the set of points on $p \in (\mathbb{R}^n)^*$ such that $p(c) \geq 0$ for all points $c \in C$. We then look at $\mathbb{Z}^n \cap C^*$. This forms a semigroup S. We consider the $\mathbb{C}$-algebra $\mathbb{C}[S]$, and define the associated toric variety to be $\mathrm{Specm}(\mathbb{C}[S])$. The faces of the cone correspond to open dense subsets of $\mathrm{Specm}(\mathbb{C}[S])$, and the open dense subset corresponding to the $\{0\}$ face is precisely the eponymous torus of the toric variety. Generalizing this construction to fans is easy: construct the affine toric variety for each cone in the face, and glue them along the open dense subsets corresponding to the intersection face. Looking at a few examples should make things clearer.

Example 1.1. Consider the cone C spanned by the vectors $(0, 1)$ and $(1, 1)$ in $\mathbb{R}^2$ (See Figure 1). The first step is to determine the dual cone C^*: if we pick the standard basis for $(\mathbb{R}^2)^*$, we get the dual cone to be the cone in Figure 2. The cone is spanned by $(1, 0)$, and $(-1, 1)$. The associated $\mathbb{C}$-algebra is isomorphic to $\mathbb{C}[x, y]$, and the associated toric variety is $\mathbb{C}^2$.

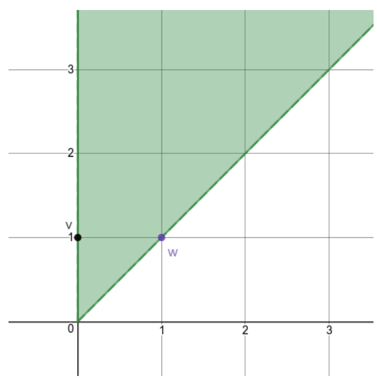

FIGURE 1. The cone spanned by $v = (0, 1)$ and $w = (1, 1)$ in $\mathbb{R}^2$.

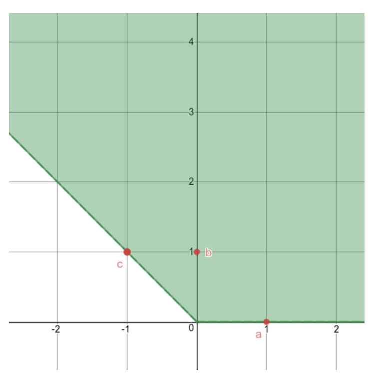

FIGURE 2. The dual cone C^*. The semigroup $C^* \cap \mathbb{Z}^2$ is spanned by $(1, 0)$, and $(-1, 1)$.

One can also build projective varieties via fans, as the following example demonstrates.

Example 1.2. Consider a fan $\mathcal{U}$ generated by three cones C_1, C_2, and C_3. The cone C_1 is generated by $(0, 1)$ and $(1, 0)$ in $\mathbb{R}^2$, the cone C_2 is generated by $(1, 0)$ and $(-1, -1)$, and the cone C_3 is generated by $(-1, -1)$ and $(0, 1)$ (see Figure 3). The coordinate rings of the affine toric variety corresponding to each of the cones are $\mathbb{C}[x, y]$, $\mathbb{C}[xy^{-1}, y^{-1}]$, and $\mathbb{C}[x^{-1}, x^{-1}y]$, where the monomials have been labelled to indicate the manner in which they're glued. It's easy to see that these are exactly the three affine open charts that cover $\mathbb{CP}^2$, so the toric variety we get from the fan $\mathcal{U}$ is $\mathbb{CP}^2$.

Now that we can construct toric varieties from fans, the next step is to construct fans from lattice polytopes in $\mathbb{R}^n$. A lattice polytope P in $\mathbb{R}^n$ is the convex hull of a finite set of points $\{v_1, \ldots, v_n\} \subset \mathbb{Z}^n$, called vertices. For each vertex v of P, we can consider the cone C_v generated by $v_i - v$, as v_i varies over all vertices on P. The collection of the dual cones C_v^* for all vertices v forms a fan, and this is the fan we associate to the polytope P. The associated toric variety shall be denoted by X_P.

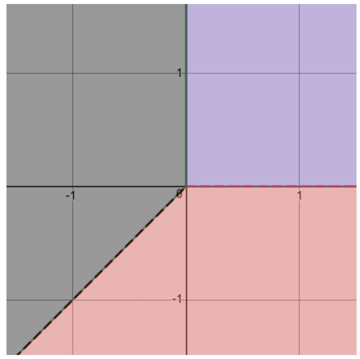

FIGURE 3. The fan whose associated toric variety is $\mathbb{CP}^2$.

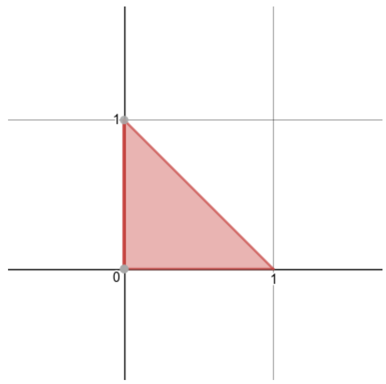

FIGURE 4. The polytope P whose associated toric variety is $\mathbb{CP}^2$.

Example 1.3. Consider the polytope P whose vertices are $v_1 = (0, 0)$, $v_2 = (0, 1)$, and $v_3 = (1, 0)$ (see Figure 4). The dual cone associated to v_1 is generated by $(0, 1)$ and $(1, 0)$. The dual cone associated to v_2 is generated by $(1, 0)$ and $(-1, -1)$. The dual cone associated to v_3 is generated by $(0, 1)$ and $(-1, -1)$. This is exactly the fan from Example 1.2. That means the associated toric variety X_P is $\mathbb{CP}^2$.

We now have a way of assigning to each lattice polytope P a toric variety X_P. A natural question to ask at this point is whether this assignment is injective, i.e. if we take two lattice polytopes P and P', such that $P \neq P'$, are the associated toric varieties non isomorphic? This is clearly false, since P and kP, where kP is a scaled version of P have the same toric varieties. The amendment we need to make to the question then is how different do P and P' need to be for the associated toric varieties to be non isomorphic. An amendment along a slightly different direction is asking whether we can put any additional structure on X_P that detects the difference between X_P and X_{kP}, i.e. something structure on X_P that detects the side lengths of P. An easier question to ask would be whether it's possible to recover P from a toric variety X such that $X_P \cong X$. This is the question we will be investigating, and to answer this we'll need to endow a toric variety with the structure of a symplectic manifold.

2. Symplectic Toric Varieties

In this and the subsequent sections, we'll be looking at toric varieties as smooth real manifolds, which means we'll be thinking about projective varieties with the classical topology, and will restrict ourselves to non-singular varieties. It's easy to detect whether the toric variety associated to a cone is non-singular or not: the affine toric variety is non-singular iff the cone is generated by a subset of a basis of $\mathbb{Z}^n$.

Definition 2.1 (Symplectic form)**.** A symplectic form ω on a real manifold M is a closed 2-form which is non-degenerate, i.e. for any v, there exists a w such that $\omega(v, w) \neq 0$.

Example 2.1 (Standard symplectic form on S^2)**.** On S^2, we can construct a symplectic structure via the standard embedding in $\mathbb{R}^3$. Given two tangent vectors v and w at a point p in S^2, $\omega(v, w)$ is given by the following formula.

$$\omega(v, w) = \langle v \times w, n_p \rangle_{\mathbb{R}^3}$$

In the above formula, $v \times w$ is the cross product of tangent vectors in $\mathbb{R}^3$, n_p is the outward pointing unit normal vector at p, and the inner product is the standard Euclidean inner product on $\mathbb{R}^3$.

One might expect to construct symplectic forms on the higher dimensional even spheres analogous to this construction, but that fails because $H^2_{\mathrm{dR}}(S^{2n}) = 0$ for $n > 1$. Recall that a symplectic form is a closed 2-form, hence an element of H^2_{dR}, and the non-degeneracy ensures that ω^n is non-degenerate as well. Since a non-degenerate top dimensional form on a compact manifold cannot be exact, that means that the second de Rham cohomology of a symplectic manifold can never be 0. The correct higher dimensional version of this construction is the standard symplectic form on $\mathbb{CP}^n$, but to construct that, we need the symplectic form on $\mathbb{C}^{n+1}$.

Example 2.2 (Symplectic structure on $\mathbb{C}^{n+1}$)**.** Let $\{z_1, \ldots, z_{n+1}\}$ be the standard coordinates on $\mathbb{C}^{n+1}$. The standard symplectic form on $\mathbb{C}^{n+1}$ is given by the following expression.

$$\omega = \frac{i}{2} \sum_{i=1}^{n+1} dz_i \wedge d\overline{z}_i$$

To get a symplectic structure on $\mathbb{CP}^n$, we need to understand how group actions behave in the symplectic category, since $\mathbb{CP}^n$ can be gotten by looking at the orbit space of $\mathbb{C}^{n+1}$ under the $\mathbb{C}^*$ action.

We will concentrate on actions by the real torus, $(S^1)^m$, to make the subsequent discussion simpler.

Definition 2.2 (Hamiltonian torus action)**.** The action of a real torus $(S^1)^m$ on a symplectic manifold (M, ω) is said to be Hamiltonian if the following conditions are satisfied.

- The action is effective, i.e. for each non-identity element $g \in (S^1)^m$, there is some point $p \in M$ moved by g, $g(p) \neq p$.
- The action preserves the symplectic form ω, i.e. for any $g \in (S^1)^m$, $\phi_g^* \omega = \omega$.
- There exists a moment map for the action, i.e. a map $\mu : M \to \mathbb{R}^m$ such that for any tangent vector $X \in T_e((S^1)^m) = \mathbb{R}^m$, the following identity holds.

$$d(\langle \mu(x), X \rangle) = (\phi_* X)^*$$

Here, $\phi_*(X)$ is the vector field on M generated by the pushforward of the vector field X on $(S^1)^m$, and $(\phi_* X)^*$ is the 1-form obtained by the map from T_pM to T_p^*M given by ω.

This definition is rather formidable, especially the third condition, so it helps to look at examples of Hamiltonian torus actions.

2.1. Examples of Hamiltonian Torus Actions.

Example 2.3 (Hamiltonian action on $\mathbb{CP}^1$)**.** Consider $\mathbb{CP}^1$, with the standard symplectic structure from Example 2.1. The action of the algebraic torus $\mathbb{C}^*$ is given by just multiplication by a complex numbers. If we restrict the action to the real torus in $\mathbb{C}^*$, i.e. the points z with norm 1, we get an action of S^1 on $\mathbb{CP}^2$. The action can be described as follows: if we think of $\mathbb{CP}^2$ sitting inside $\mathbb{R}^3$, such that 0 and ∞ are the north and the south pole, the action of θ on the point (x, y, z) sends it to $(x \cos(\theta), y \sin(\theta), z)$. This action is certainly effective, because every group element moves at least some point. To check that the action preserves the form ω, write down the form in cylindrical coordinates (h, θ), which becomes $\omega = d\theta \wedge dh$. Since the action sends (h, θ) to $(h, \theta + \alpha)$, this action clearly preserves the symplectic form. The last thing we need to show is that there is a moment map. One possible candidate is the height function, i.e. in cylindrical coordinates, the map that sends (h, θ) to h. The tangent vector we'll pick on $T_e(S^1)$ will be a fixed non-zero vector; any other vector will just be a multiple of this. This corresponds to the vector field $\frac{\partial}{\partial \theta}$ on $\mathbb{CP}^2$. The corresponding 1-form is exactly dh, since $\omega\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial h}\right) = 1$. This shows that the left and right hand side agree in the definition of the Hamiltonian action, so this action is indeed Hamiltonian.

One thing to note about the previous example is that the image of the moment map is the set $[-1, 1] \subset \mathbb{R}$. Coincidentally, the toric variety $\mathbb{CP}^1$ is obtained exactly by taking the toric variety associated to the polytope $[-1, 1]$. We'll soon see that this is not a coincidence, and that the moment map of the Hamiltonian torus action gives us a way of recovering the polytope from a toric variety.

Example 2.4 (Moment map for S^1 action on $\mathbb{C}^n$)**.** Recall that we were interested in the $\mathbb{C}^*$ action on $\mathbb{C}^n$. If we restrict the action to $S^1 \subset \mathbb{C}^*$ consisting of norm 1 elements, we get an effective action. To see why this action preserves the symplectic form $\frac{i}{2} \sum_k dz_k \wedge d\overline{z}_k$, we can rewrite it in polar coordinates as $\sum_k r_k dr_k \wedge d\theta_k$. Clearly, the action of S^1 just adds a constant to all the θ_k components, which means it preserves the symplectic form. To find the moment map μ, let's determine what $d\mu$ must be. Since μ is the moment map, $d\mu$ must equal $(\phi_* X)^*$, where

$\phi_* X$ is the vector field given by $\sum_k \frac{\partial}{\partial \theta_k}$. Since the symplectic form is $\sum_k r_k dr_k \wedge d\theta_k$, $\left(\sum_k \frac{\partial}{\partial \theta_k}\right)^* = -\sum_k r_k dr_k$. That means $d\mu = -\sum_k r_k dr_k$, and $\mu = -\frac{\sum_k r_k^2}{2} + C$, for any constant C.

In the above example, let $C = \frac{1}{2}$, which ensures $\mu^{-1}(0)$ is the standard $2n-1$ sphere. We have the S^1 action on S^{2n+1}, and the quotient space is exactly $\mathbb{CP}^{n-1}$. We would like to somehow like to get a symplectic structure on $\mathbb{CP}^{n-1}$ using this link to $\mathbb{C}^n$. Note that on $\mathbb{C}^n$, the standard symplectic form is also a Kähler form, and there is an associated Kähler metric. That descends to a Riemannian metric on the subspace S^{2n-1}, and that descends to a metric on $\mathbb{CP}^{n-1}$, which is Kähler. The associated Kähler form is a symplectic form on $\mathbb{CP}^{n-1}$. That Kähler form actually turns out to be what is called the Fubini-Study form. This gives us a symplectic structure on all $\mathbb{CP}^n$, and also their products.

Example 2.5 (Moment map for S^1 action on $\mathbb{CP}^1$ with the Kähler form)**.** Consider $\mathbb{CP}^1$ with the Kähler form. Consider the open set U_0 where the first homogeneous coordinate $z_0 = 1$. In these coordinates, the Kähler form is given by the following formula.

$$\begin{aligned} \omega &= i\partial\overline{\partial} \log\left(1 + z_1\overline{z}_1\right) \\ &= \frac{i}{(1 + z_1\overline{z}_1)^2} dz_1 \wedge d\overline{z}_1 \end{aligned}$$

The action of the torus S^1 sends z to $e^{i\theta}z$, which clearly preserves the symplectic form. For the unit tangent vector on S^1, the induced vector field on $\mathbb{CP}^1$ is the following.

$$iz_1 \frac{\partial}{\partial z_1} - i\overline{z}_1 \frac{\partial}{\partial \overline{z}_1}$$

The associated 1-form can be obtained via ω, and is the following.

$$-\frac{\overline{z}_1 dz_1 + z_1 d\overline{z}_1}{(1 + z_1\overline{z}_1)^2}$$

To find the moment map, we need to find a function μ such that $d\mu$ is the above 1-form. One such function is the following.

$$\mu = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{1 + z_1\overline{z}_1}\right)$$

This is the formula for the moment for the moment map that also generalizes for higher dimensional projective spaces. If we write it out in homogeneous coordinates, the moment map looks like the following.

$$\mu([z_0 : z_1]) = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{z_0\overline{z}_0 + z_1\overline{z}_1}\right)$$

Example 2.6 (Moment map for $S^1 \times S^1$ action on $\mathbb{CP}^2$)**.** As in the previous example, let's focus on the open set U_0 where the first homogeneous coordinate is 1. The action of $S^1 \times S^1$ on this open set is given by the following expression.

$$\phi\left((\theta_1, \theta_2), [1 : z_1 : z_2]\right) = [1 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$$

In this chart, the Kähler form is given by the following expression.

$$\begin{aligned} \omega &= i\partial\overline{\partial} \log\left(1 + z_1\overline{z}_1 + z_2\overline{z}_2\right) \\ &= \frac{i}{\left(1 + z_1\overline{z}_1 + z_2\overline{z}_2\right)^2}\left(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 - z_1\overline{z}_2 dz_2 \wedge d\overline{z}_1 - z_2\overline{z}_1 dz_1 \wedge d\overline{z}_2\right) \end{aligned}$$

Note that now the tangent space of the group acting on $\mathbb{CP}^2$ is two dimensional, which means we'll need to look at two vector fields: the one induced by $\frac{d}{d\theta_1}$, and the one induced by $\frac{d}{d\theta_2}$. The vector fields they induce are the following.

$$\begin{aligned} \frac{d}{d\theta_1} &\mapsto iz_1 \frac{\partial}{\partial z_1} - i\overline{z}_1 \frac{\partial}{\partial \overline{z}_1} \\ \frac{d}{d\theta_2} &\mapsto iz_1 \frac{\partial}{\partial z_2} - i\overline{z}_2 \frac{\partial}{\partial \overline{z}_2} \end{aligned}$$

If we now look at the associated 1-forms $d\mu_1$ and $d\mu_2$, the functions μ_1 and μ_2 will be the moment maps. Those turn out to be the following.

$$\mu_1([1:z_1:z_2]) = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{1+z_1\overline{z}_1+z_2\overline{z}_2}\right)$$
$$\mu_2([1:z_1:z_2]) = -\frac{1}{2}\left(\frac{z_2\overline{z}_2}{1+z_1\overline{z}_1+z_2\overline{z}_2}\right)$$

Thus the global moment map $\mu : \mathbb{CP}^2 \to \mathbb{R}^2$ is given by the following expression.

$$\mu([z_0:z_1:z_2]) = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{z_0\overline{z}_0+z_1\overline{z}_1+z_2\overline{z}_2}, \frac{z_2\overline{z}_2}{z_0\overline{z}_0+z_1\overline{z}_1+z_2\overline{z}_2}\right)$$

We can now look at the image of this map as $[z_0 : z_1 : z_2]$ varies over $\mathbb{CP}^2$ (See Figure 5). Notice that this is exactly a scaled version of the polytope P whose toric variety is $\mathbb{CP}^2$.

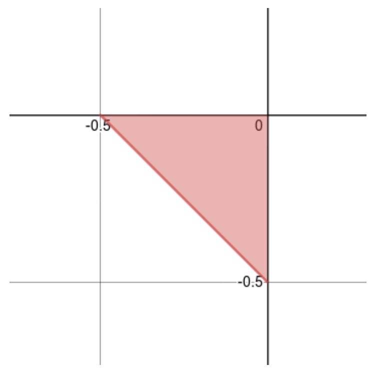

FIGURE 5. The image of the moment map for $\mathbb{CP}^2$.

Example 2.7 (Moment map for $(S^1)^n$ action on $\mathbb{CP}^n$)**.** Proceeding in a manner similar to the previous two examples, we can write down an explicit formula for the action and the Kähler form. If we replicate the previous calculations, we get the moment map to be the following.

$$\mu([z_0:\cdots:z_n]) = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}, \cdots, \frac{z_n\overline{z}_n}{z_0\overline{z}_0+\cdots+z_n\overline{z}_n}\right)$$

Example 2.8 (Moment map for $S^1 \times S^1$ action on $\mathbb{CP}^1 \times \mathbb{CP}^1$)**.** We have an action of $S^1 \times S^1$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by the following expression.

$$\phi((\theta_1,\theta_2),([1:z_1],[1:w_1])) = ([1:e^{i\theta_1}z_1],[1:e^{i\theta_2}w_1])$$

We also have a Kähler form on $\mathbb{CP}^1 \times \mathbb{CP}^1$ because the product of Kähler manifolds is Kähler. The Kähler form is the following.

$$\omega = \frac{idz_1 \wedge d\overline{z}_1}{(1+z_1\overline{z}_1)^2} + \frac{idw_1 \wedge d\overline{w}_1}{(1+w_1\overline{w}_1)^2}$$

The vectors fields on $\mathbb{CP}^1 \times \mathbb{CP}^1$ corresponding to $\frac{d}{d\theta_1}$ and $\frac{d}{d\theta_2}$ are the following.

$$\frac{d}{d\theta_1} \mapsto i\left(z_1\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial \overline{z}_1}\right)$$
$$\frac{d}{d\theta_2} \mapsto i\left(w_1\frac{\partial}{\partial w_1} - \overline{w}_1\frac{\partial}{\partial \overline{w}_1}\right)$$

The associated 1-forms are analogous to the Example 2.5. That means if (μ_1,μ_2) is a moment map for this toric variety, then we get the expressions for $d\mu_1$ and $d\mu_2$ to be the following.

$$d\mu_1 = -\frac{\overline{z}_1dz_1 + z_1d\overline{z}_1}{(1+z_1\overline{z}_1)^2}$$
$$d\mu_2 = -\frac{\overline{w}_1dw_1 + w_1d\overline{w}_1}{(1+w_1\overline{w}_1)^2}$$

We thus get the moment map to be precisely the following.

$$\mu(([z_0 : z_1], [w_0 : w_1])) = -\frac{1}{2}\left(\frac{z_1\overline{z}_1}{z_0\overline{z}_0 + z_1\overline{z}_1}, \frac{w_1\overline{w}_1}{w_0\overline{w}_0 + w_1\overline{w}_1}\right)$$

The image of this moment map is the polytope whose associated toric variety is $\mathbb{CP}^1 \times \mathbb{CP}^1$.

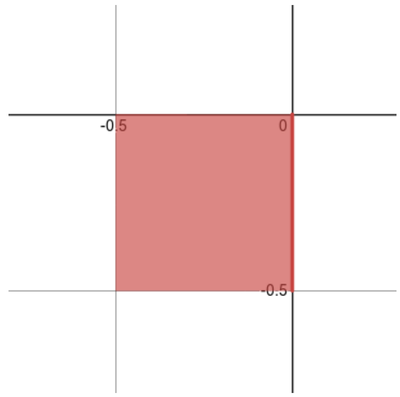

FIGURE 6. The image of the moment map for $\mathbb{CP}^1 \times \mathbb{CP}^1$.

All these examples make it reasonable to conjecture that the image of a moment map is a lattice polytope (up to scaling), such that the associated toric variety is isomorphic to the original toric variety. That is in fact true, and we'll sketch out a part of the proof in the next section. We'll also look at a more interesting example.

3. The Toric Variety associated to the moment polytope

Before we go on to sketch a proof of our claim, we shall state it precisely.

Claim 1. *Given a symplectic manifold (M, ω), with $(S^1)^m$ acting on M via a Hamiltonian action, the image of the moment map μ is a polytope P in $\mathbb{R}^n$, called the moment polytope. This polytope is a lattice polytope up to scaling, and the toric variety X_P associated to this polytope is diffeomorphic to M.*

The first step in proving the claim is showing that the image of the moment map μ is actually a polytope. In fact, a much stronger result is true, which states that the images form what are called Delzant polytopes.

Definition 3.1 (Delzant polytopes)**.** A Delzant polytope P in $\mathbb{R}^n$ is a polytope satisfying the following properties.

- n edges meet at each vertex.
- For any pair of vertices v_i and v_j, $v_i - v_j$ is a multiple of a vector with rational coordinates.
- For any vertex v, the collection $\{v_i - v\}$ as v_i varies over all vertices can be scaled to form a basis of $\mathbb{Z}^n$.

The result, due to Delzant, states the following.

Theorem 2 (Delzant [3])**.** *There is a bijective correspondence between toric symplectic manifolds, and Delzant polytopes. The map in one direction is given by the following.*

$$(M^{2n}, \omega, (S^1)^n, \mu) \mapsto \mu(M)$$

Notice that the third criterion for a Delzant polytope is exactly the condition we need to ensure that the toric variety associated to a polytope is smooth. This hints that the bijection can be reversed via the toric variety construction.

The next step in proving the claim is showing that toric varieties which actually come from lattice polytopes can be given a symplectic structure, and that the image of the associated moment map is the original lattice polytope, up to scaling. We'll sketch this construction with some details (this construction is [2]).

Consider a lattice polytope P in $\mathbb{R}^n$. It turns out that the homogeneous coordinate of the projective variety X_P is generated by the lattice points in P, i.e. $P \cap \mathbb{Z}^n$. Suppose P has k lattice points in it, the set of which we'll denote by A.

$$\begin{aligned} A &:= K \cap \mathbb{Z}^n \\ &= \left\{\lambda^{(0)}, \ldots, \lambda^{(k-1)}\right\} \end{aligned}$$

These k lattice points give an equivariant embedding of X_P into $\mathbb{CP}^{k-1}$, since one can associate monomials to these points that generate the homogeneous coordinate ring of X_P. The action of $(S^1)^n$ on $\mathbb{CP}^{k-1}$ is given by the following expression.

$$(t_1, \ldots, t_n) \cdot [z_0 : \cdots : z_{k-1}] \mapsto \left[t^{\lambda^{(0)}} z_0 : \cdots : t^{\lambda^{(k-1)}} z_{k-1}\right]$$

It is from this ambient space that we inherit a symplectic structure. Recall that $\mathbb{CP}^{k-1}$ has a Kähler form, and the restriction of the Kähler form to a smooth subvariety is also Kähler, hence non-degenerate. This restricted Kähler form is the symplectic structure we want on X_P. Furthermore, note that the Kähler form is invariant under the torus action as well, since X_P is equivariantly embedded in $\mathbb{CP}^{k-1}$. One can write down the moment map from $\mathbb{CP}^{k-1}$ to $\mathbb{R}^n$ for this action.

$$\mu([z_0 : \cdots : z_{k-1}]) = \frac{\sum_{i=0}^{k-1} \lambda^{(i)} |z_i|^2}{\sum_{i=0}^{k-1} |z_i|^2}$$

One can check that the image of the moment map μ from $\mathbb{CP}^{k-1}$ is the polytope A. All we need to do now is to show that the image of the restriction to X_P is still A. One does that by showing that every vertex of A lies in the image of $\mu |_{X_P}$. Since the image of μ is a polytope by Delzant's theorem, that proves the result. This concludes the second step of the proof of the claim.

Ideally, the third step of the proof would be to show that all smooth projective toric varieties come from lattice polytopes, but this result is only known to be true for dimensions 1 and 2 (see [1]). However, all these results suggest that the moment map is useful in determining the polytope a toric variety comes from. We shall explore its utility in one last example, an example where we really don't know from what lattice polytope a toric variety is coming from.

Example 3.1 (The moment map for $S^1 \times S^1$ action on the Hirzebruch surfaces)**.** The n^{th} Hirzebruch surface $\mathcal{H}_n$ is defined to be the projective bundle over $\mathbb{CP}^1$ corresponding to the vector bundle $O(0) \oplus O(n)$. In the case when $n = 0$, the surface $\mathcal{H}_n$ is just $\mathbb{CP}^1 \times \mathbb{CP}^1$, and we've seen what lattice polytope that comes from. To see what lattice polytope $\mathcal{H}_n$ comes from, for $n \neq 0$, we will put a symplectic structure on $\mathcal{H}_n$, and then compute the image of the moment map. That will be the polytope we want.

The first step is to determine an embedding of $\mathcal{H}_n$ in some projective space. It turns out that $\mathcal{H}_n$ embeds inside $\mathbb{CP}^1 \times \mathbb{CP}^2$ as the vanishing set of the following polynomial.

$$\mathcal{H}_n = Z(x_0^n y_1 - x_1^n y_2)$$

Here, the coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^2$ are given by $([x_0 : x_1], [y_0 : y_1 : y_2])$. But just an embedding isn't enough: we want to make sure it's an equivariant embedding, i.e. the action of $S^1 \times S^1$ on $\mathcal{H}_n$ extends to the ambient space $\mathbb{CP}^1 \times \mathbb{CP}^2$. If one plays around with the torus action a bit, one can see that the following action works.

$$(\theta_1, \theta_2) \cdot ([x_0 : x_1], [y_0 : y_1 : y_2]) = ([x_0 : e^{i\theta_1} x_1], [y_0 : e^{i(n\theta_1 + \theta_2)} y_1 : e^{i\theta_2} y_2])$$

The next thing we need is the symplectic form, which comes from the Kähler form on $\mathbb{CP}^1 \times \mathbb{CP}^2$. In local coordinates where $x_0 = 1$ and $y_0 = 1$, it's given by the following expression.

$$\omega = i \left(\frac{dx_1 \wedge d\overline{x}_1}{(1 + x_1 \overline{x}_1)^2} + \frac{dy_1 \wedge d\overline{y}_1 + dy_2 \wedge d\overline{y}_2 - y_1 \overline{y}_2 dy_2 \wedge d\overline{y}_1 - y_2 \overline{y}_1 dy_1 \wedge d\overline{y}_2}{(1 + y_1 \overline{y}_1 + y_2 \overline{y}_2)^2} \right)$$

Checking that the action preserves the symplectic form is easy, and essentially the same check as in previous examples. Constructing the moment map μ for the action also follows the same recipe as the previous example, and isn't very hard. We'll skip the actual calculations for the sake of brevity, and just write down the expression for the moment map we end up with.

$$\mu([x_0 : x_1], [y_0 : y_1 : y_2]) = -\frac{1}{2} \begin{pmatrix} \dfrac{x_1 \overline{x}_1}{x_0 \overline{x}_0 + x_1 \overline{x}_1} + n \left(\dfrac{y_1 \overline{y}_1}{y_0 \overline{y}_0 + y_1 \overline{y}_1 + y_2 \overline{y}_2} \right) \\ \dfrac{y_1 \overline{y}_1 + y_2 \overline{y}_2}{y_0 \overline{y}_0 + y_1 \overline{y}_1 + y_2 \overline{y}_2} \end{pmatrix}$$

The final step is to restrict the map μ to $\mathcal{H}_n$, and in local coordinates, that corresponds to setting $y_1 = x_1^n y_2$. If we do that, and compute the image of this map in $\mathbb{R}^2$, we get the following sequence of polytopes, for n ranging from 1 to 3 (See Figures 7, 8, and 9).

Since the Hirzebruch surfaces are smooth projective toric surfaces, we can conclude that the moment polytopes of these surfaces are exactly the associated lattice polytopes.

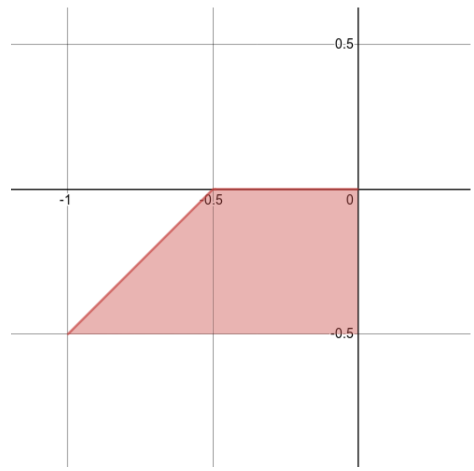

FIGURE 7. The moment polytope for $\mathcal{H}_1$.

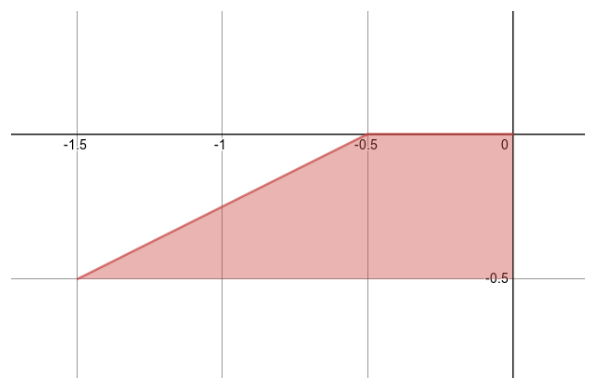

FIGURE 8. The moment polytope for $\mathcal{H}_2$.

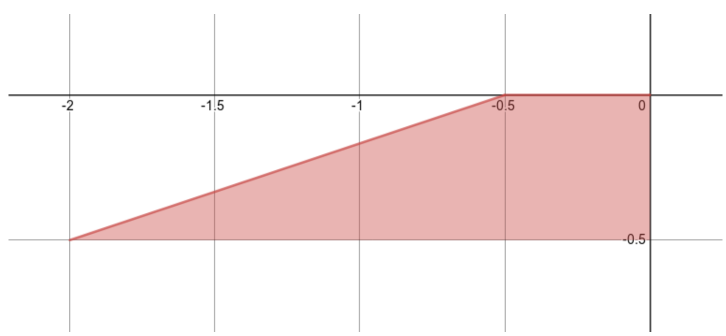

FIGURE 9. The moment polytope for $\mathcal{H}_3$.

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